L^p-Theory of Venttsel BVPs with Discontinuous Data

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We will present some very recent results regarding the regularity and solvability theory of second-order elliptic equations with discontinuous coefficients coupled with boundary conditions of Venttsel type, given in terms of second-order differential operators with discontinuous data. Precisely, we deal with the problem

$$\begin{cases} \mathcal{L}u := -a^{ij}(x)D_iD_ju + b^i(x)D_iu + c(x)u = f(x), \\ \mathcal{B}u := -\alpha^{ij}(x)d_id_ju + \beta^i(x)d_iu + \beta^0(x)\partial_{\mathbf{n}}u + \gamma(x)u = g(x) \end{cases}$$

over a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, with $C^{1,1}$-smooth boundary. Here $\mathbf{n}$ is the unit outward normal to $\partial\Omega$ and du stands for the tangential gradient of u, $d_i = D_i - \mathbf{n}_i\mathbf{n}_jD_j$. The solutions are understood in a *strong* sense, that is, these belong to suitable Sobolev spaces of twice weakly differentiable functions and satisfy the equations above at *almost every* point x. The natural space $V_{p,q}(\Omega)$ to study that problem consists of all functions $u \in W^2_p(\Omega)$ with traces in $W^2_q(\partial\Omega)$, where the exponents p and q satisfy $1 < p \leq \frac{nq}{n-1} < p^*$, $q > 1$, with p^* being the Sobolev conjugate of p. The principal coefficients a^{ij} and α^{ij} of the *uniformly elliptic* operators $\mathcal{L}$ and $\mathcal{B}$ are supposed to be *VMO*-functions in Ω and $\partial\Omega$, respectively, while optimal Lebesgue or Orlicz integrability is required on the lower-order coefficients.

We derive, first of all, an *a priori* estimate for the $V_{p,q}(\Omega)$-norm of any strong solution in terms of $\|f\|_{L^p(\Omega)}$ and $\|g\|_{L^q(\partial\Omega)}$. Based on this, the *elliptic regularization property* is obtained for the couple $(\mathcal{L}, \mathcal{B})$ in the framework of the Sobolev spaces and, under additional assumptions on the vector field $(\beta^1, \ldots, \beta^n)$ and the coefficients c and γ, *strong solvability* is proved for the Venttsel BVP in $V_{p,q}(\Omega)$ for *all* admissible values of p and q.

Strong solvability of the *quasilinear* Venttsel problem

$$\begin{cases} -a^{ij}(x,u)D_iD_ju + a(x,u,Du) = 0 & \text{a.e. in } \Omega, \\ -\alpha^{ij}(x,u)d_id_ju + \alpha(x,u,Du) = 0 & \text{a.e. on } \partial\Omega \end{cases}$$

with *discontinuous* coefficients will be discussed as well.

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