Exercise session 1

Fractional Calculus

Exercise 1

Let $\alpha > 0$, $\beta > -1$ *and* $a < b$. *Calculate:*

1) *the left-hand fractional integral* $\left(I_{a+}^{\alpha} f\right)(x)$ *for* $x > a$, *where* $f(t) = (t-a)^{\beta}$;
2) *the right-hand fractional integral* $\left(I_{b-}^{\alpha} f\right)(x)$ *for* $x < b$, *where* $f(t) = (b-t)^{\beta}$.

Exercise 2

Let $\alpha, \beta > 0$, $a < b$ *and* $f \in L^1(a, b)$. *Prove that for any* $x \in (a, b)$

$$(I_{a+}^{\beta}(I_{a+}^{\alpha} f))(x) = (I_{a+}^{\alpha+\beta} f)(x).$$

Exercise 3

Solve, at least formally, the integral Abel equation

$$f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{\phi(t)}{(x-t)^{1-\alpha}}\, dt, \;\; x \in [a,b], \;\; 0 < \alpha < 1.$$

Remark

Recall that $f \in AC([a,b])$ *if and only if there exists a function* $f' \in L^1(a,b)$ *such that for any* $x \in [a,b]$

$$f(x) = f(a) + \int_a^x f'(t)dt.$$

Exercise 4

Let $\alpha \geq 1$ *and* $f \in L^1(a,b)$*. Show that* $I^{\alpha}_{a+} f \in AC([a,b])$*.*

Exercise 5

Let $0 < \alpha < 1$ *and* $f \in AC([a, b])$*. Prove that* $I_{a+}^{1-\alpha} f \in AC([a, b])$ *and*

$$(I_{a+}^{1-\alpha} f)(x) = \frac{1}{\Gamma(2-\alpha)} \left[f(a)(x-a)^{1-\alpha} + \int_a^x f'(t)(x-t)^{1-\alpha} dt \right]. \quad (1)$$

Exercise 6

Let $f \in AC([a, b])$. Show that Abel's equation with $0 < \alpha < 1$ is solvable in $L_1(a, b)$ and its solution can be represented as

$$\phi(x) = \frac{1}{\Gamma(1-\alpha)} \left[\frac{f(a)}{(x-a)^\alpha} + \int_a^x \frac{f'(s)}{(x-s)^\alpha} \, ds \right]. \tag{2}$$

Exercise 7

Let $\left(D^{\alpha}_{a+}f\right)(x) = \dfrac{1}{\Gamma(1-\alpha)}\dfrac{d}{dx}\left(\int_a^x \dfrac{f(t)}{(x-t)^{\alpha}}\,dt\right)$ *be left-hand side fractional derivative (Riemann-Liouville fractional derivative). Observe that, in general,*

$$\left(D^{\alpha}_{a+}f\right)(x) = \frac{d}{dx}(I^{1-\alpha}_{a+}f)(x)$$

Prove that for $0 < \alpha < 1$ *and* $f \in AC([a,b])$ *such derivative exists almost everywhere, and*

$$\left(D^{\alpha}_{a+}f\right)(x) \quad = \quad \frac{1}{\Gamma(1-\alpha)}\left[\frac{f(a)}{(x-a)^{\alpha}} + \int_a^x \frac{f'(t)}{(x-t)^{\alpha}}\,dt\right].$$

Exercise 8

Let $f(x) = (x-a)^{-\mu}$, $0 < \mu < 1$. *Prove that for* $\alpha + \mu < 1$

$$\left(D_{a+}^{\alpha} f\right)(x) = \frac{\Gamma(1-\mu)}{\Gamma(1-\mu-\alpha)} \frac{1}{(x-a)^{\mu+\alpha}},$$

and that $\left(D_{a+}^{\alpha} f\right)(x) = 0$ *if* $f(x) = (x-a)^{\alpha-1}$.

Exercise 9

a) *Let* $\phi \in L_1([a,b])$. *Prove that for any* $0 < \alpha < 1$ $\left(D^{\alpha}_{a+} I^{\alpha}_{a+}\phi\right)(x) = \phi(x)$, *a.e. with respect to the Lebesgue measure.*

b) *Let* ϕ *can be represented as the fractional integral of order* α *of some function from* $L_1([a,b])$. *Prove that for any* $x \in [a,b]$

$$\left(I^{\alpha}_{a+} D^{\alpha}_{a+}\phi\right)(x) = \phi(x).$$