Exercise session 3

Parameter Estimation for models driven by a fractional Brownian motion

Ergodic theorem for Gaussian processes

Let $\{X_n\}_{n\geq 0}$ be a discrete-time stationary zero-mean Gaussian stochastic process such that $EX_0X_n \to 0$ as $n \to +\infty$. Then $\{X_n\}_{n\geq 0}$ is ergodic and, in particular, for any function $h : \mathbb{R}^k \to \mathbb{R}$,

$$\frac{1}{n}\sum_{j=0}^{n-1} h(X_j, \ldots, X_{j+k-1}) \to E[h(X_0, \ldots, X_{k-1})] \quad \text{almost surely.}$$

Continuous mapping theorem

Let $(X_n^{(1)}, \ldots, X_n^{(k)}) \to (X^{(1)}, \ldots, X^{(k)})$ in distribution,almost surely or in probability and let $g : \mathbb{R}^k \to \mathbb{R}$ be a continuous function. Then $g(X_n^{(1)}, \ldots, X_n^{(k)}) \to g(X^{(1)}, \ldots, X^{(k)})$ respectively in distribution, almost surely or in probability.

Exercise 1

Let $X_n = X_0 + \sigma B_n^H$, where $X_0 \in \mathbb{R}$, $\sigma > 0$ and $H \in (0,1)$. Show that

$$\widehat{\sigma}_n^2 = QV_n := \frac{1}{n}\sum_{k=0}^{n-1}(X_{k+1} - X_k)^2$$

is an unbiased strongly consistent estimator for σ^2.

Observe that $X_{k+1} - X_k = \sigma I_{k,1}^H$. We have already shown that $I_{k,1}^H$ is stationary and $EI_{0,1}^H I_{k,1}^H \to 0$ by Ex. 3 of Session 2, thus we can use the ergodic theorem for Gaussian processes to conclude that

$$QV_n = \frac{1}{n}\sum_{k=0}^{n-1}(X_{k+1} - X_k)^2 = \frac{\sigma^2}{n}\sum_{k=0}^{n-1}(I_{k,1}^H)^2 \to \sigma^2 E[(I_{0,1}^K)^2] = \sigma^2$$

almost surely, thus it is strongly consistent. To show that $\widehat{\sigma}_n^2$ is unbiased, just observe that $E[(I_{k,1}^H)^2] = 1$ by Ex. 1 of Session 2 and then

$$E[\widehat{\sigma}_n^2] = \frac{\sigma^2}{n}\sum_{k=0}^{n-1} E[(I_{k,1}^H)^2] = \sigma^2.$$

Exercise 2

Consider the model X_n defined before with unknown σ and H.

a) *Show that*

$$V_{1,n} = \frac{1}{n-1}\sum_{k=1}^{n-1}(X_{k+1} - 2X_k + X_{k-1})^2 \to (4 - 2^{2H})\sigma^2 \quad \textit{almost surely.}$$

b) *Use the previous result to construct a strongly consistent estimator $\widehat{H}_n^{(1)}$ of the Hurst parameter H in the form $f(\frac{V_{1,n}}{QV_n})$ for some continuous function f.*

Observe that $X_{k+1} - 2X_k + X_{k-1} = \sigma(I^H_{k,1} - I^H_{k-1,1})$. Again, we can use the ergodic theorem for Gaussian processes to achieve

$$V_{1,n} = \frac{\sigma^2}{n-1}\sum_{k=1}^{n-1}(I^H_{k,1} - I^H_{k-1,1})^2 \to \sigma^2 E(I^H_{1,1} - I^H_{0,1})^2 = \sigma^2 E(B^H_2 - 2B^H_1)^2$$

Now observe

$$
\begin{aligned}
E(B_2^H - 2B_1^H)^2 &= E(B_2^H)^2 - 4EB_2^H B_1^H + 4E(B_1^H)^2 \\
&= 2^{2H} - 2(2^{2H} + 1 - 1) + 4 = 4 - 2^{2H},
\end{aligned}
$$

concluding a).

For b), recall that QV_n is a strongly consistent estimator of σ^2. Continuous mapping theorem suggests to consider $\widehat{H}_n^{(1)}$ such that $V_{1,n} = QV_n(4 - 2^{2\widehat{H}_n^{(1)}})$, i.e.

$$
\widehat{H}_n^{(1)} = \frac{1}{2}\log_2\left(4 - \frac{V_{1,n}}{QV_n}\right).
$$

To verify that $\widehat{H}_n^{(1)}$ is a strongly consistent estimator of H, use continuous mapping theorem to observe that

$$
\begin{aligned}
\widehat{H}_n^{(1)} &\to \frac{1}{2}\log_2\left(4 - \frac{\sigma^2(4 - 2^{2H})}{\sigma^2}\right) \\
&= \frac{1}{2}\log_2\left(4 - 4 + 2^{2H}\right) = H.
\end{aligned}
$$

Exercise 3

Consider $\delta > 0$ and the model $X_n = X_0 + \sigma B^H_{\delta n}$ with unknown σ and H.

a) *Show that $\widehat{H}^{(1)}_n$ is still a strongly consistent estimator of H;*

b) *Using QV_n and $\widehat{H}^{(1)}_n$, construct a strongly consistent estimator $\widehat{\sigma}^2_n$ of σ^2.*

Set, for simplicity, $t_k = k\delta$. Let us evaluate the limit of QV_n. Observing that $t_{k+1} = t_k + \delta$ we have $X_{k+1} - X_k = \sigma I^H_{\delta k,\delta}$. Again, the process $I^H_{\delta k,\delta}$ is ergodic (with the same arguments as in Ex. 3 of Session 2), thus we have

$$QV_n \to \sigma^2 E(I^H_{0,\delta})^2 = \sigma^2 \delta^{2H}.$$

With the same argument we have

$$V_{1,n} \to \sigma^2 E(I^H_{\delta,\delta} - I^H_{0,\delta})^2 = \sigma^2 E(B^H_{2\delta} - 2B^H_{\delta})^2.$$

This time we use self-similarity of B_t^H to conclude that

$$V_{1,n} \to \delta^{2H}\sigma^2 E(B_2^H - 2B_1^H)^2 = \delta^{2H}\sigma^2(4 - 2^{2H}).$$

By continuous mapping theorem we have

$$\widehat{H}_n^{(1)} \to \frac{1}{2}\log_2\left(4 - \frac{\sigma^2\delta^{2H}(4-2^{2H})}{\sigma^2\delta^{2H}}\right) = H.$$

Concerning point b), since $QV_n \to \sigma^2\delta^{2H}$, the continuous mapping theorem suggest to use $\widehat{\sigma}_n^2$ such that $QV_n = \widehat{\sigma}_n^2\delta^{2\widehat{H}_n^{(1)}}$, that is to say

$$\widehat{\sigma}_n^2 = QV_n\delta^{-2\widehat{H}_n^{(1)}}.$$

Indeed, the function $f(\sigma^2, H) = \sigma^2\delta^{-2H}$ is continuous and then, by continuous mapping theorem, we have

$$\widehat{\sigma}_n^2 \to \sigma^2\delta^{2H}\delta^{-2H} = \sigma^2.$$

Exercise 4

Consider the model $X_t = X_0 + \sigma B_t^H$ *for* $t \in [0, T]$ *with unknown* σ *and* H. *Consider* $N \in \mathbb{N}$ *with* $N > 1$ *and define* $t_k^N = \frac{T}{N-1}k$ *for* $0 \leq k \leq N-1$. *Consider* $X_k^N = X_{t_k^N}$ *and define*

$$QV_N = \frac{1}{N}\sum_{k=0}^{N-1}(X_{k+1}^N - X_k^N)^2$$

and

$$V_{1,N} = \frac{1}{N-1}\sum_{k=0}^{N-1}(X_{k+1}^N - 2X_k^N + X_{k+1}^N)^2.$$

a) *Show that* $(N-1)^{2H} QV_N \xrightarrow{d} T^{2H}\sigma^2$ *and* $(N-1)^{2H} V_{1,N} \xrightarrow{d} (4 - 2^{2H}) T^{2H}\sigma^2$;

b) *Show that* $\widehat{H}_N^{(1)}$ *is still a consistent estimator of* H.

Let us evaluate the limit of QV_N. Observing that $t^N_{k+1} = t^N_k + \frac{T}{N-1}$ we have $X^N_{k+1} - X^N_k = \sigma I^H_{\frac{T}{N-1}k, \frac{T}{N-1}}$. Using self-similarity properties of $I^H_{t,s}$ from Ex. 4 of Session 2 we have

$$QV_N \overset{d}{=} \frac{\sigma^2}{N}\left(\frac{T}{N-1}\right)^{2H} \sum_{k=0}^{N-1} (I^H_{k,1})^2.$$

Thus, by ergodic theorem, we conclude that

$$(N-1)^{2H} QV_N \overset{d}{=} \frac{T^{2H}\sigma^2}{N} \sum_{k=0}^{N-1} (I^H_{k,1})^2 \to T^{2H}\sigma^2.$$

Again, by self-similarity, we have

$$\begin{aligned} V_{1,N} &= \frac{\sigma^2}{N-1} \sum_{k=1}^{N-1} \left(I^H_{\frac{T}{N-1}k, \frac{T}{N-1}} - I^H_{\frac{T}{N-1}(k-1), \frac{T}{N-1}}\right)^2 \\ &\overset{d}{=} \frac{\sigma^2}{N-1}\left(\frac{T}{N-1}\right)^{2H} \sum_{k=1}^{N-1} (I^H_{k,1} - I^H_{k-1,1})^2. \end{aligned}$$

Again, by ergodic theorem, we have

$$(N-1)^{2H} V_{1,N} \xrightarrow{d} T^{2H}\sigma^2(4-2^{2H}).$$

To show point b), let us first observe, by continuous mapping theorem,

$$\begin{aligned}
\widehat{H}_N^{(1)} &= \frac{1}{2}\log_2\left(4-\frac{V_{1,N}}{QV_N}\right) \\
&= \frac{1}{2}\log_2\left(4-\frac{(N-1)^{2H}V_{1,N}}{(N-1)^{2H}QV_N}\right) \\
&\xrightarrow{d} \frac{1}{2}\log_2\left(4-\frac{T^{2H}\sigma^2(4-2^{2H})}{T^{2H}\sigma^2}\right) = H.
\end{aligned}$$

However, being H a constant value (and not a random variable), in such specific case convergence in distribution implies convergence in probability.

Exercise 5

Let $X_n = X_0 + \sigma B^H_n$ *with unknown* $\sigma > 0$ *and* $H \in (0,1)$*.*

a) *Show that*

$$V_{2,n} = \frac{1}{n-1}\sum_{k=1}^{n-1}(X_{k+1} - X_{k-1})^2 \to 2^{2H}\sigma^2 \textit{ almost surely;}$$

b) *Use the previous result to construct a strongly consistent estimator* $\widehat{H}^{(2)}_n$ *of the Hurst parameter* H *in the form* $f(V_{2,n}/QV_n)$ *for some continuous function* f*;*

c) *Consider* $\delta > 0$ *and* $X_n = X_0 + \sigma B^H_{\delta n}$ *with unknown* σ *and* H*. Prove that* $\widehat{H}^{(2)}_n$ *is still a strongly consistent estimator of* H*.*

d) Consider $X_t = X_0 + \sigma B_t^H$ for $t \in [0, T]$ with unknown σ and H. Fix $N \in \mathbb{N}$ with $N > 1$ and define $t_k^N = \frac{T}{N-1}k$ for $0 \leq k \leq N-1$. Finally, consider $X_k^N = X_{t_k^N}$. Define

$$V_{2,N} = \frac{1}{N-1}\sum_{k=1}^{N-1}(X_{k+1}^N - X_{k-1}^N)^2 \to 2^{2H}\sigma^2$$

Prove that $\widehat{H}_N^{(2)}$ is still a consistent estimator of H.

To solve item a), observe that $X_{k+1} - X_{k-1} = \sigma(I_{k,1}^H + I_{k-1,1}^H)$ and then use the ergodic theorem for Gaussian processes to achieve

$$V_{2,n} = \frac{\sigma^2}{n-1}\sum_{k=1}^{n-1}(I_{k,1}^H + I_{k-1,1}^H)^2 \to \sigma^2(I_{1,1}^H + I_{0,1}^H)^2 = \sigma^2 E(B_2^H)^2 = \sigma^2 2^{2H}.$$

For item b), continuous mapping theorem suggests to set $\widehat{H}_n^{(2)}$ such that $V_{2,n} = QV_n 2^{2\widehat{H}_n^{(2)}}$, i.e.

$$\widehat{H}_n^{(2)} = \frac{1}{2}\log_2\left(\frac{V_{2,n}}{QV_n}\right) \to \frac{1}{2}\log_2\left(\frac{\sigma^2 2^{2H}}{\sigma^2}\right) = H.$$

Concerning item c), set $t_k = k\delta$ and recall that, by Ex. 3, $QV_n \to \delta^{2H}\sigma^2$. On the other hand, using the fact that $t_{k+1} = t_k + \delta$, we have, using ergodicity of $I_{k\delta,\delta}$,

$$V_{2,n} = \frac{\sigma^2}{n-1}\sum_{k=1}^{n-1}(I_{\delta k,\delta}^H + I_{\delta(k-1),\delta}^H)^2 \to \sigma^2 E(B_{2\delta}^H)^2 = \sigma^2 2^{2H}\delta^{2H}.$$

Hence, by the continuous mapping theorem,

$$\widehat{H}_n^{(2)} = \frac{1}{2}\log_2\left(\frac{V_{2,n}}{QV_n}\right) \to \frac{1}{2}\log_2\left(\frac{\sigma^2 2^{2H}\delta^{2H}}{\sigma^2\delta^{2H}}\right) = H.$$

Now let us work with item d). To do this, recall from Ex. 4 that $(N-1)^{2H}QV_N \xrightarrow{d} T^{2H}\sigma^2$. Concerning $V_{2,N}$, we have, setting $\delta_N = \frac{T}{N-1}$,

$$\begin{aligned}(N-1)^{2H}V_{2,N} &= \frac{\sigma^2}{N-1}(N-1)^{2H}\sum_{k=1}^{N-1}(I_{\delta_N k,\delta_N} + I_{\delta_N(k-1),\delta_N})^2 \\ &\stackrel{d}{=} \sigma^2 T^{2H}\frac{1}{N-1}\sum_{k=1}^{N-1}(I_{k,1}+I_{k-1,1})^2 \\ &\to \sigma^2 T^{2H}E(B_2^H)^2 = 2^{2H}T^{2H}\sigma^2,\end{aligned}$$

where we also used the ergodic theorem for Gaussian processes.

Hence, by continuous mapping theorem

$$\begin{aligned}
\widehat{H}_N^{(2)} &= \frac{1}{2}\log_2\left(\frac{V_{2,N}}{QV_N}\right)\\
&= \frac{1}{2}\log_2\left(\frac{(N-1)^{2H}V_{2,N}}{(N-1)^{2H}QV_N}\right)\\
&\xrightarrow{d} \frac{1}{2}\log_2\left(\frac{T^{2H}\sigma^2 2^{2H}}{T^{2H}\sigma^2}\right) = H.
\end{aligned}$$

Being H a constant value, this is enough to imply consistency of $\widehat{H}_N^{(2)}$.

Exercise 6

Let $X_n = X_0 + \sigma(B_n^H - B_{n-1}^H)$ *with unknown* $\sigma > 0$ *and* $H \in (0,1)$*. Denote* $\overline{X}_n = \frac{1}{n}\sum_{k=1}^n X_k$ *and define the family of functionals* $S : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ *as*

$$S_n(x) = \sqrt{\frac{1}{n}\sum_{k=1}^{n}(x_k - \bar{x}_n)^2}.$$

Show that $S_n^2(X)$ *is a strongly consistent estimator of* σ^2*.*

Hints

Set $U_n = \frac{1}{n}\sum_{k=1}^n (I_{k,1}^H)^2$, $Y_n = \frac{B_n^H}{n}$ and $F(U,Y) = U - Y^2$. Observe that F is a continuous function.

Observe that

$$\bar{X}_n = \frac{1}{n}\sum_{k=1}^{n} X_k = X_0 + \frac{\sigma}{n}\sum_{k=1}^{n}(B_k^H - B_{k-1}) = X_0 + \sigma\frac{B_n^H}{n}.$$

Thus

$$\begin{aligned} S_n^2(X) &= \frac{1}{n}\sum_{k=1}^{n}\left(X_0 + \sigma(B_k^H - B_{k-1}^H) - X_0 - \sigma\frac{B_n^H}{n}\right)^2 \\ &= \sigma^2\frac{1}{n}\left(\sum_{k=1}^{n}(B_k^H - B_{k-1}^H)^2 - 2\sum_{k=1}^{n}\frac{B_n^H(B_k^H - B_{k-1}^H)}{n} + \sum_{k=1}^{n}\left(\frac{B_n^H}{n}\right)^2\right) \\ &= \sigma^2\left(\frac{1}{n}\sum_{k=1}^{n}(I_{k,1}^H)^2 - 2\frac{B_n^H}{n^2}\sum_{k=1}^{n}(B_k^H - B_{k-1}^H) + \left(\frac{B_n^H}{n}\right)^2\right) \\ &= \sigma^2\left(\frac{1}{n}\sum_{k=0}^{n-1}(I_{k,1}^H)^2 - \left(\frac{B_n^H}{n}\right)^2\right) \\ &= \sigma^2 F(U_n, Y_n), \end{aligned}$$

where $F(U, Y) = U - Y^2$, $U_n = \frac{1}{n}\sum_{k=1}^{n}(I_{k,1}^H)^2$ and $Y_n = \frac{B_n^H}{n}$.

By continuous mapping theorem, we only have to study the limits of U_n and Y_n. Concerning U_n, by ergodic theorem, we have

$$U_n \to E[(I_{0,1}^H)^2] = 1.$$

Concerning Y_n, we can rewrite it in terms of $I_{k,1}^H$ as

$$Y_n = \frac{1}{n} \sum_{k=0}^{n-1} I_{k,1}^H \to E[I_{0,1}^H] = 0,$$

where the limit follows from the ergodic theorem. Thus, continuous mapping theorem implies that

$$S_n^2(X) \to \sigma^2 F(1,0) = \sigma^2.$$

Exercise 7

Let X_n *be the same as in the previous exercise. Let* $R_n : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ *as*

$$R_n(X) = \max_{1 \le k \le n}(k(\bar{X}_k - \bar{X}_n)) - \min_{1 \le k \le n}(k(\bar{X}_k - \bar{X}_n))$$

and $\Phi : C([0,1]) \to \mathbb{R}$ *defined as*

$$\Phi(f) = \max_{t \in [0,t]}(f(t) - tf(1)) - \min_{t \in [0,t]}(f(t) - tf(1)).$$

Show that $n^{-H} R_n(X) \to \sigma \Phi(B^H)$ *in distribution.*

Hints

Use self-similarity of B_t^H and recall that for any $f \in C([0,1])$ it holds

$$\max_{\frac{1}{n} \le \frac{k}{n} \le 1} f\left(\frac{k}{n}\right) \to \max_{t \in [0,1]} f(1), \; n \to +\infty,$$

and the same holds for the minimum.

Recall that $\bar{X}_n = X_0 + \sigma \frac{B_n^H}{n}$, thus

$$k(\overline{X}_k - \overline{X}_n) = \sigma \left(B_k^H - \frac{k}{n} B_n^H \right) = \sigma \left(B_{\frac{kn}{n}}^H - \frac{k}{n} B_n^H \right) \stackrel{d}{=} \sigma n^H \left(B_{\frac{k}{n}}^H - \frac{k}{n} B_1^H \right).$$

In particular this implies

$$\begin{aligned} R_n(X) &\stackrel{d}{=} \sigma n^H \left(\max_{1 \leq k \leq n} \left(B_{\frac{k}{n}}^H - \frac{k}{n} B_1^H \right) - \min_{1 \leq k \leq n} \left(B_{\frac{k}{n}}^H - \frac{k}{n} B_1^H \right) \right) \\ &= \sigma n^H \left(\max_{\frac{1}{n} \leq \frac{k}{n} \leq 1} \left(B_{\frac{k}{n}}^H - \frac{k}{n} B_1^H \right) - \min_{\frac{1}{n} \leq \frac{k}{n} \leq 1} \left(B_{\frac{k}{n}}^H - \frac{k}{n} B_1^H \right) \right). \end{aligned}$$

Being $f(t) = B_t^H - tB_1^H$ a (a.s.) continuous function, we can use the hint to conclude that

$$n^{-H} R_n(X) \xrightarrow{d} \sigma \left(\max_{0 \leq t \leq 1} \left(B_t^H - tB_1^H \right) - \min_{0 \leq t \leq 1} \left(B_t^H - tB_1^H \right) \right) = \sigma \Phi(B^H).$$

Exercise 8

Let X_n be the same as in Ex. 6 and consider $S_n, R_n : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ defined respectively in Ex. 6 and 7 and define $(R/S)_n(X) = R_n(X)/S_n(X)$.

a) *Show that*

$$2^{-nH} \frac{(R/S)_{2^{2n}}(X)}{(R/S)_{2^n}(X)} \xrightarrow{d} 1.$$

b) *Use the previous result to construct a consistent estimator $\widehat{H}_n$ of H.*

Slutsky's Theorem

Let $(X_n^{(1)}, \dots, X_n^{(k)})$ be a sequence of random variables such that $(X_n^{(1)}, \dots, X_n^{(k)}) \xrightarrow{d} (X^{(1)}, \dots, X^{(k)})$, where $(X^{(1)}, \dots, X^{(k)})$ is a random variable, ad let $(\widehat{\Theta}_n^{(1)}, \dots, \widehat{\Theta}_n^{(h)})$ be a sequence of random variables such that $(\widehat{\Theta}_n^{(1)}, \dots, \widehat{\Theta}_n^{(h)}) \xrightarrow{\mathbb{P}} (\widehat{\Theta}^{(1)}, \dots, \widehat{\Theta}^{(h)})$, where $(\widehat{\Theta}^{(1)}, \dots, \widehat{\Theta}^{(h)})$ is a constant. Let $g : \mathbb{R}^k \times \mathbb{R}^h \to \mathbb{R}$ be a continuous function. Then

$$g(X_n^{(1)}, \dots, X_n^{(k)}, \widehat{\Theta}_n^{(1)}, \dots, \widehat{\Theta}_n^{(h)}) \xrightarrow{d} g(X^{(1)}, \dots, X^{(k)}, \widehat{\Theta}^{(1)}, \dots, \widehat{\Theta}^{(h)})$$

To prove a), first recall that, by Ex. 6 $S_n(X) \to \sigma$ almost surely, thus also in probability. On the other hand , by Ex. 7, $n^{-H} R_n(X) \xrightarrow{d} \sigma \Phi(B^H)$. Thus, by Slutsky's theorem (which can be applied since $\sigma > 0$),

$$\frac{n^{-H} R_n(X)}{S_n(X)} \xrightarrow{d} \Phi(B^H).$$

By continuous mapping theorem we have

$$2^{-nH} \frac{(R/S)_{2^{2n}}(X)}{(R/S)_{2^n}(X)} = \frac{2^{-2nH}(R/S)_{2^{2n}}(X)}{2^{-nH}(R/S)_{2^n}(X)} \xrightarrow{d} \frac{\Phi(B^H)}{\Phi(B^H)} = 1.$$

Concerning point b), continuous mapping theorem suggest to choose $\widehat{H}_n$ such that

$$2^{-n\widehat{H}_n}\frac{(R/S)_{2^{2n}}(X)}{(R/S)_{2^n}(X)} = 1,$$

i.e.

$$\widehat{H}_n = \frac{1}{n}\log_2\left(\frac{(R/S)_{2^{2n}}(X)}{(R/S)_{2^n}(X)}\right).$$

Indeed, by continuous mapping theorem, we have

$$\begin{aligned}\widehat{H}_n &= \frac{1}{n}\log_2\left(2^{nH}2^{-nH}\frac{(R/S)_{2^{2n}}(X)}{(R/S)_{2^n}(X)}\right)\\ &= H + \frac{1}{n}\log_2\left(2^{-nH}\frac{(R/S)_{2^{2n}}(X)}{(R/S)_{2^n}(X)}\right) \xrightarrow{d} H.\end{aligned}$$

Convergence in probability follows as H is a constant value.

Exercise 9

Let $X_n = X_0 + \theta n + \sigma B_n^H$ *with* $X_0 \in \mathbb{R}$ *and unknown* $\theta \in \mathbb{R}$, $\sigma > 0$ *and* $H \in (0, 1)$.

a) *Show that* $\widehat{\theta}_n = \frac{X_n}{n}$ *is a strongly consistent estimator of* θ*;*

b) *Show that* $QV_n \to \theta^2 + \sigma^2$*;*

c) *Use point b) to construct a strongly consistent estimator* $\widehat{\sigma}_n^2$ *of* σ^2*;*

d) *Construct a strongly consistent estimator* $\widehat{H}_n$ *of* H *of the form* $f(V_{1,n}/\widehat{\sigma}_n^2)$ *for some continuous function* f*.*

To show item a), observe that

$$\frac{X_n}{n} = \frac{X_0}{n} + \theta + \frac{B_n^H}{n} \to \theta,$$

where the convergence is justified by the ergodic theorem, rewriting $B_n^H = \sum_{k=0}^n I_{k,1}^H$.

To show item b), observe that $X_k - X_{k-1} = \theta + \sigma I_{k-1,1}^H$ and then

$$\begin{aligned} QV_n &= \frac{1}{n-1}\sum_{k=1}^n (\theta + \sigma I_{k-1,1}^H)^2 \\ &= \theta^2 + \frac{1}{n-1}\sum_{k=1}^n (2\theta\sigma I_{k-1,1}^H + \sigma^2 (I_{k-1,1}^H)^2). \end{aligned}$$

By the ergodic theorem, we have

$$QV_n \to \theta^2 + E(2\theta\sigma I_{0,1}^H + \sigma^2 (I_{0,1}^H)^2) = \theta^2 + \sigma^2.$$

Concerning point c), continuous mapping theorem suggests to set

$$\widehat{\sigma}_n^2 = QV_n - \widehat{\theta}_n^2 \to \sigma^2 + \theta^2 - \theta^2 = \sigma^2.$$

To solve point d), observe that $X_{k+1} - 2X_k + X_{k-1} = I^H_{k,1} + I^H_{k-1,1}$, as in the case $\theta = 0$. Thus, again, $V_{1,n} \to \sigma^2(4 - 2^{2H})$. Continuous mapping theorem suggests to set

$$\begin{aligned}
\widehat{H}_n &= \frac{1}{2}\log_2\left(4 - \frac{V_{1,n}}{\widehat{\sigma}_n^2}\right) \\
&\to \frac{1}{2}\log_2\left(4 - \frac{\sigma^2(4-2^{2H})}{\sigma^2}\right) = H.
\end{aligned}$$

Exercise session 4

Parameter Estimation for models driven by a fractional Brownian motion: practical session

Exercise

Implement the estimators discussed in the previous Session of Exercises. You can use any calculus environment (MATLAB, R, Mathematica, also C or Fortran if you prefer). In particular, for any estimator, test it with a suitable choice of parameters for $n = 50$, 100, 200, 500, 1000. Use 5000 samples. In particular evaluate the following quantities:

- The average of each estimator
- The bias of each estimator (i.e. the difference between the average and the actual parameter to estimate);
- The variance of each estimator;
- The mean square error of each estimator (i.e. the sum of the variance and the square of the bias).

Recall that we will use R in the exercise session.

In R, use the `somebm` package. Precisely, there is the function `fbm` that generates a skeleton in $[0, 1]$ of n nodes for a fractional Brownian motion of Hurst index H. The syntax is given by:

```
fbm(hurst=H, n=nodes)
```

Then, one can use self-similarity of the fBm to extend (or reduce) the interval to $[0, T]$.