# Exercise Session #1

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- The DF $q$ is increasing so, in particular, $F_q$ is non decreasing in $\mathbb{R}$.
- The DF $q$ is continuous so, in particular, $F_q$ is right continuous in $\mathbb{R}$.
- The DF $q$ is a function $q : [0,1] \to [0,1]$ such that $q(0) = 0$ and $q(1) = 1$, so

$$lim_{t \to 0} F_q(t) = 0 \quad \text{and} \quad lim_{t \to 1} F_q(t) = 1.$$

- The same for $F_Q$.

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- $n > 1$

$$Q(u_1, u_2) = u_1 u_2 [1 + (1-u_1)(1-u_2)], \; u_1, u_2 \in [0,1].$$

FGM copula

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- $\bar{F}_{i:n}(t)$ is a distorted distribution with

$$\bar{q}_{i:n}(u) = \sum_{j=0}^{i-1} \binom{n}{j} (1-u)^j u^{n-j}$$

$$q_{i:n}(t) = \sum_{j=i}^{n} \binom{n}{j} u^j (1-u)^{n-j}$$

$$\bar{q}_{2:3}(u) = \sum_{j=0}^{1} \binom{3}{j} \underbrace{(1-u)^j}_{F} \underbrace{u^{3-j}}_{= \bar{F}}$$

$$= \binom{3}{0}(1-u)^0 u^{3-0} + \binom{3}{1}(1-u)^1 u^{3-1} =$$

$$= u^3 + 3(1-u)u^2 = u^3 + 3u^2 - 3u^3$$

$$= 3u^2 - 2u^3$$

- For $X_{2:3}$ we have

$$\bar{q}_{2:3}(u) = q_{2:3}(u) = 3u^2 + 2u^3$$

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- The mixture survival function

$$\bar{F}_{\mathbf{p}}(t) = p_1 \bar{F}_1(t) + \cdots + p_n \bar{F}_n(t),$$

where $= \mathbf{p}(p_1, \ldots, p_n), p_i \geq 0$ and $p_1, \cdots, p_n = 1$.

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where $= \mathbf{p}(p_1, \ldots, p_n), p_i \geq 0$ and $p_1, \cdots, p_n = 1$.

- It is a generalized distorted distribution with

$$\bar{Q}(u_1, \ldots, u_n) = p_1 u_1 + \cdots + p_n u_n, \; u_i \in [0, 1].$$

$$F_p(t) = \frac{1}{2}\bar{F}(t) + \frac{1}{2}\bar{F}^2(t)$$

- The distortion function of a fifty-fifty mixture of $\bar{F}$ and $\bar{F}^2$ is

$$\bar{Q}(u,u) = \bar{q}(u) = \frac{1}{2}u + \frac{1}{2}u^2 = \frac{1}{2}u(1+u).$$

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that is a GDD from $F_1, F_2$ with $Q_{2:2} = C$.

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- Clayton copula

$$C(u_1, u_2) = \frac{u_1 u_2}{u_1 + u_2 - u_1 u_2}.$$

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- So

$$Q_{2:2}(u_1, u_2) = \frac{u_1 u_2}{u_1 + u_2 - u_1 u_2}.$$

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- So

$$q_{2:2}(u) = \frac{u^2}{2u - u^2} = \frac{u}{2-u}.$$

Clayton copula ID components

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- $\bar{Q}_{2:2}(u_1, u_2) = u_1 + u_2 - u_1 u_2$
- If we compute, for example, $\bar{Q}_{2:2}(1, u_2)$

$$\bar{Q}_{2:2}(1, u_2) = 1 + u_2 - x_2 = 1.$$

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- If we compute, for example, $\bar{Q}_{2:2}(1, u_2)$

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- $\bar{Q}_{2:2}(1, u_2) \neq u_2 \Rightarrow \bar{Q}_{2:2}$ is not a copula.

$C(0,y) = C(x,0) = 0$

$C(x,1) = x$

$C(1,y) = y$

Exercise 9. Compare the order statistics $X_{2:3}$ and $X_{3:3}$ (IID case).

$$F_{3:3}(t) = F^3(t) \Rightarrow \varphi_{3:3}(u) = u^3$$

$$\overline{F}_{3:3}(t) = 1 - F_{3:3}(t) = 1 - F^3(t) = 1 - (1 - \overline{F}(t))^3$$

$$\Rightarrow \overline{\varphi}_{3:3}(u) = 1 - (1 - u)^3 = 3u - 3u^2 + u^3$$

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- $q_{3:3}(u) = u^3$ and $\bar{q}_{3:3}(u) = 3u - 3u^2 + u^3$.
- $\bar{q}_{2:3}(u) = q_{2:3}(u) = 3u^2 - 2u^3$.
- ST order

$$q_{3:3}(u) \le q_{2:3}(u) \Rightarrow X_{2:3} \le_{ST} X_{3:3}$$

$$\bar{q}_{3:3} \ge \bar{q}_{2:3}(u)$$

$$u^3 \le 3u^2 - 2u^3 \Leftrightarrow u \le 3 - 2u \Leftrightarrow 3 - 3u \ge 0$$

$$3(1-u) \ge 0 \Leftrightarrow 1 - u \ge 0 \quad \text{ok!} \quad \forall u \in (0,1)$$

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- $\bar{q}_{2:3}(u) = q_{2:3}(u) = 3u^2 - 2u^3$.
- ST order
$$q_{3:3}(u) \leq q_{2:3}(u) \Rightarrow X_{2:3} \leq_{ST} X_{3:3}$$
- RHR order
$$\frac{q_{3:3}(u)}{q_{2:3}(u)} = \frac{u^3}{3u^2 - 2u^3} = \frac{u}{3 - 2u} \quad \text{increases in} \quad [0, 1]$$

$\Rightarrow X_{2:3} \leq_{RHR} X_{3:3}$

- $\bar{q}'_{2:3}(u) = 6u - 6u^2$ and $\bar{q}'_{3:3}(u) = 3u^2 - 6u + 3.$

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- LR order

$$\frac{\bar{q}'_{3:3}(u)}{\bar{q}'_{2:3}(u)} = \frac{3u^2 - 6u + 3}{6u - 6u^2} = \frac{u^2 - 2u + 1}{2u - 2u^2} \quad \text{decreases in} \quad [0, 1]$$

$\Rightarrow X_{2:3} \leq_{LR} X_{3:3}.$

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- HR order

$$\frac{\bar{q}_{3:3}(u)}{\bar{q}_{2:3}(u)} = \frac{3u - 3u^2 + u^3}{3u^2 - 2u^3} = \frac{3 - 3u + u^2}{3u - 2u^2} \quad \text{decreases in} \quad [0,1]$$

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$\Rightarrow X_{2:3} \leq_{HR} X_{3:3}$

- MRL order

$$X_{2:3} \leq_{HR} X_{3:3} \Rightarrow X_{2:3} \leq_{MRL} X_{3:3}$$

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- We have to compute

$$\alpha_{2:3}(u) = u\frac{\bar{q}'_{2:3}(u)}{\bar{q}_{2:3}} = \frac{6u^2 - 6u^3}{3u^2 - 2u^3} = \frac{6 - 6u}{3u - 2u}$$

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- $\bar{q}_{2:3}(uv) = 3u^2v^2 - 2u^3v^3$
  $\bar{q}_{2:3}(u) = 3u^2 - 2u^3$
  $\bar{q}_{2:3}(v) = 3v^2 - 2v^3$ } $(3u^2 - 2u^3)(3v^2 - 2v^3)$ //

$$\bar{q}_{2:3}(uv) \leq \bar{q}_{2:3}(u)\bar{q}_{2:3}(v),\ \forall u, v \in [0,1]$$

$\Rightarrow$ NBU class is preserved by $X_{2:3}$.

- $\bar{q}_{2:3}(u^c) = 3u^{2c} - 2u^{3c}$
  $(\bar{q}_{2:3}(u))^c = (3u^2 - 2u^3)^c$

$$\bar{q}_{2:3}(u^c) \geq (\bar{q}_{2:3}(u))^c, \ \forall u, c \in [0, 1]$$

$\Rightarrow$ IFRA class is preserved by $X_{2:3}$.

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- DFR class is preserved by the mixture $\Rightarrow$ DFRA class is preserved.

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- In the IID case

$$\bar{Q}_{2:2}(u, u) = \bar{q}_{2:2}(u) = 2u - u^2 \to \bar{q}'_{2:2}(u) = 2 - 2u.$$

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- IFR class is preserved by $X_{2:2} \Rightarrow$ NBU class is preserved by $X_{2:2}$.

$$\overline{q}_{1:2}(uv) \leq \overline{q}_{2:2}(u)\,\overline{q}_{2:2}(v)$$

$$2uv - u^2v^2 \leq (2u - u^2)(2v - v^2)$$

$$\iff 2uv - u^2v^2 \leq 4uv - 2uv^2 - 2u^2v + u^2v^2$$

$$\iff 2uv - 2uv^2 - 2u^2v + 2u^2v^2 \geq 0$$

$$\iff 1 - v - u + uv \geq 0$$

$$\iff uv \geq v + u - 1$$

ok! $\forall u, v \in (0,1)$

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- $\bar{Q}_{2:2}(u_1 v_1, u_2 v_2) = u_1 v_1 + u_2 v_2 - u_1 v_1 u_2 v_2$
  $\bar{Q}_{2:2}(u_1, u_2) = u_1 + u_2 - u_1 u_2$
  $\bar{Q}_{2:2}(v_1, v_2) = v_1 + v_2 - v_1 v_2$ $\Big\}$ $(u_1 + u_2 - u_1 u_2)(v_1 + v_2 - v_1 v_2)$

$$\bar{Q}_{2:2}(u_1 v_1, u_2 v_2) \leq \bar{Q}_{2:2}(u_1, u_2)\bar{Q}_{2:2}(v_1, v_2), \ \forall u_i, v_i \in [0, 1]$$

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  $\bar{Q}_{2:2}(u_1, u_2) = u_1 + u_2 - u_1 u_2$
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$$\bar{Q}_{2:2}(u_1 v_1, u_2 v_2) \leq \bar{Q}_{2:2}(u_1, u_2)\bar{Q}_{2:2}(v_1, v_2), \; \forall u_i, v_i \in [0, 1]$$

$\Rightarrow$ NBU class is preserved by $X_{2:3}$.

- NB: ALL the systems with indipendent components preserve the NBU class.

Exercise 14. Study which classes are preserved by $X_{2:2}$ in the ID case for a copula $C$.

$$\overline{Q}_{2:2}(u_1, u_2) = u_1 + u_2 - \hat{C}(u_1, u_2)$$

$$\hat{C}(u_1, u_2) = \frac{u_1 u_2}{u_1 + u_2 - u_1 u_2}$$

Clayton survival copula

$$\overline{Q}_{2:2}(u_1, u_2) = \overline{q}_{2:2}(u) = 2u - \hat{C}(u)$$

$$\hat{C}(u) = \frac{u^2}{2u - u^2} = \frac{u}{2-u}$$

$$\overline{q}_{2:2}(u) = 2u - \frac{u}{2-u} = \frac{2u(2-u) - u}{2-u} = \frac{4u - 2u^2 - u}{2-u}$$

$$= \frac{3u - 2u^2}{2-u}$$

## Exercise 14. Study which classes are preserved by $X_{2:2}$ in the ID case for a copula $C$.

- Dependent component

$$\bar{Q}_{2:2}(u_1, u_2) = u_1 + u_2 - \hat{C}(u_1, u_2)$$

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- Clayton survival copula in the ID case

$$\hat{C}(u, u) = \frac{u}{2 - u}.$$

## Exercise 14. Study which classes are preserved by $X_{2:2}$ in the ID case for a copula $C$.

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$$\bar{Q}_{2:2}(u, u) = \bar{q}_{2:2}(u) = 2u - \hat{C}(u, u).$$

- Clayton survival copula in the ID case

$$\hat{C}(u, u) = \frac{u}{2 - u}.$$

- So

$$\bar{q}_{2:2}(u) = 2u - \frac{u}{2-u} = \frac{3u - 2u^2}{2 - u}$$

- The derivative

$$\bar{q}'_{2:2}(u) = \frac{6 - 8u + 2u^2}{(2-u)^2}$$

$$\bar{q}'_{2:2}(u) = \frac{(3-4u)(2-u) + (3u - 2u^2)}{(2-u)^2}$$

- The derivative

$$\bar{q}'_{2:2}(u) = \frac{6 - 8u + 2u^2}{(2-u)^2}$$

- We have to compute

$$\alpha_{2:2}(u) = u\frac{\bar{q}'(u)}{\bar{q}} = \frac{6 - 8u + 2u^2}{6 - 7u + 2u^2}.$$

$$\alpha_{2:2}(u) = u \, \frac{6-8u+2u^2}{(2-u)^2} \cdot \frac{2-u}{3u-2u^2}$$

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- $\alpha_{2:2}(u)$ is decreasing for $u \in [0,1] \Rightarrow$
  IFR class is preserved by $X_{2:2}$ in ID case for Clayton copula.

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  IFR class is preserved by $X_{2:2}$ in ID case for Clayton copula.

- IFR class is preserved by $X_{2:2}$ in ID case for Clayton copula $\Rightarrow$
  NBU class is preserved by $X_{2:2}$ in ID case for Clayton copula.

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- IFR class is preserved by $X_{2:2}$ in ID case for Clayton copula $\Rightarrow$
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