Exercise Session #2

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Stochastic Models and Complex Systems
Summer School
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Exercises

1. Determine the minimal path and minimal cut sets of a coherent system with four components.
2. Compute the reliability of a coherent system with four components in the general case.
3. Compute the reliability of a coherent system with four components in the IND case.
4. Compute the reliability of a coherent system with four components in the ID case.
5. Compute the reliability of a coherent system with four components in the IID case.
6. Compute the reliability of a plane with four engines, two in each wing, that can fly if at least one engine works in each wing.
7. Compute the minimal and maximal signatures of a system with four components.

Exercises

8. Check an arrow in the figures for the ST, HR and LR orders of systems with IID components.
9. Check a no arrow in the figures for the ST, HR and LR orders of systems with IID components.
10. Check if $X_i \leq_{HR} X_{2:2}$ holds for IND components.
11. Check if $X_i \leq_{HR} X_{2:2}$ holds for IND HR-ordered components.
12. Check if $X_i \leq_{HR} X_{2:2}$ holds for dependent components with the Clayton copula in the slides.
13. Check an arrow in the tables and figure for the ST and HR orders of systems with IND components.
14. Check if the IFR class is preserved in a system with four IID components.

Exercise 1. Determine the minimal path and minimal cut sets of a coherent system with four components.

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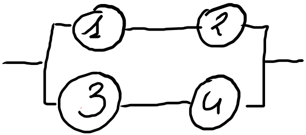

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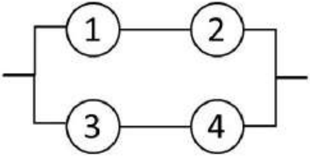

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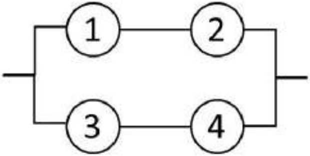

- Minimal path sets

$$P_1 = \{1, 2\} \qquad P_2 = \{3, 4\}$$

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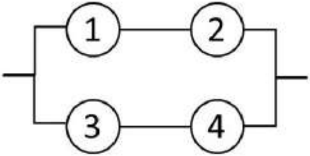

- Minimal path sets

$$P_1 = \{1, 2\} \qquad P_2 = \{3, 4\}$$

- Minimal cut sets

$$C_1 = \{1, 3\} \qquad C_2 = \{1, 4\} \qquad C_3 = \{2, 3\} \qquad C_4 = \{2, 4\}$$

Exercise 2. Compute the reliability of a coherent system with four components in the general case.

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$$\bar{F}_T(t) = \sum_{i=1}^{2} \bar{F}_{P_i}(t) - \bar{F}_{P_1 \cup P_2}$$

$$\bar{F}_T(t) = \bar{F}_{\{1,2\}} + \bar{F}_{\{3,4\}} - \bar{F}_{\{1,2,3,4\}}$$

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- $\bar{F}_T(t) = \hat{C}(\bar{F}_1(t), \bar{F}_2(t), 1, 1) + \hat{C}(1, 1, \bar{F}_3(t), \bar{F}_4(t)) - \hat{C}(\bar{F}_1(t), \bar{F}_2(t), \bar{F}_3(t), \bar{F}_4(t))$

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- $\bar{F}_T(t) = \hat{C}(\bar{F}_1(t), \bar{F}_2(t), 1, 1) + \hat{C}(1, 1, \bar{F}_3(t), \bar{F}_4(t)) - \hat{C}(\bar{F}_1(t), \bar{F}_2(t), \bar{F}_3(t), \bar{F}_4(t))$

- $\bar{F}_T(t) = \bar{Q}(\bar{F}_1(t), \bar{F}_2(t), \bar{F}_3(t), \bar{F}_4(t))$, with

$$\bar{Q}(u_1, u_2, u_3, u_4) = \hat{C}(u_1, u_2, 1, 1) + \hat{C}(1, 1, u_3, u_4) - \hat{C}(u_1, u_2, u_3, u_4)$$

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$$\bar{Q}(u_1, u_2, u_3, u_4) = \hat{C}(u_1, u_2, 1, 1) + \hat{C}(1, 1, u_3, u_4) - \hat{C}(u_1, u_2, u_3, u_4)$$

In IND case

$C(u_1, \dots u_n) = u_1 \cdot \dots \cdot u_n$

- The component are IND, so

$$\bar{Q}(u_1, u_2, u_3, u_4) = u_1 u_2 + u_3 u_4 - u_1 u_2 u_3 u_4$$

Exercise 4. Compute the reliability of a coherent system with four components in the ID case.

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- $T = max(min(X_1, X_2), min(X_3, X_4))$
- $\bar{F}_T(t) = \bar{Q}(\bar{F}(t), \bar{F}(t), \bar{F}(t), \bar{F}(t)) = \bar{q}(\bar{F}(t))$, with

$$\bar{q}(u) = \hat{C}(u, u, 1, 1) + \hat{C}(1, 1, u, u) - \hat{C}(u, u, u, u)$$

Exercise 5. Compute the reliability of a coherent system with four components in the IID case.

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- $T = max(min(X_1, X_2), min(X_3, X_4))$
- $\bar{F}_T(t) = \bar{q}(\bar{F}(t))$, with

$$\bar{q}(u) = \hat{C}(u, u, 1, 1) + \hat{C}(1, 1, u, u) - \hat{C}(u, u, u, u)$$

Exercise 5. Compute the reliability of a coherent system with four components in the IID case.

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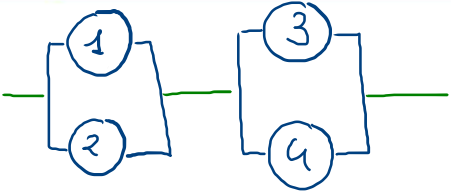
$$\bar{q}(u) = \hat{C}(u, u, 1, 1) + \hat{C}(1, 1, u, u) - \hat{C}(u, u, u, u)$$

- The components are IID, so

$\mu_1 \ \mu_2 \ \mu_3 \ \mu_4$

$$\bar{q}(u) = 2u^2 - u^4 \quad \text{and} \quad a = (0, 2, 0, -1)$$

Exercise 6. Compute the reliability of a plane with four engines, two in each wing, that can fly if at least one engine works in each wing.



$$T = \min\left(\max(X_1, X_2), \max(X_3, X_4)\right)$$

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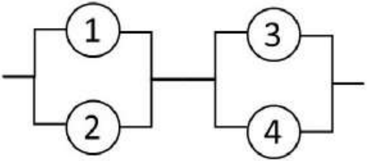

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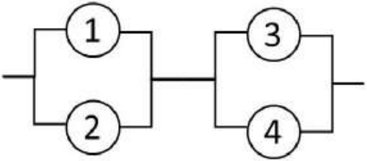

- Minimal path sets

$$P_1 = \{1, 3\} \qquad P_2 = \{1, 4\} \qquad P_3 = \{2, 3\} \qquad P_4 = \{2, 4\}$$

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$$\overline{F}_T(t) = \sum_{i=1}^{4} \overline{F}_{P_i}(t) - \sum_{i<j} \overline{F}_{P_i \cup P_j} + \sum_{i<j<s} \overline{F}_{P_i \cup P_j \cup P_s} - \overline{F}_{P_1 \cup P_2 \cup P_3 \cup P_4}$$

Here I write F_{P_i} as P_i

$$P_1 + P_2 + P_3 + P_4 - P_1 \cup P_2 - P_1 \cup P_3 - P_1 \cup P_4 - P_2 \cup P_3$$
$$- P_2 \cup P_4 - P_3 \cup P_4 + P_1 \cup P_2 \cup P_3 + P_1 \cup P_3 \cup P_4 +$$
$$P_1 \cup P_2 \cup P_4 + P_2 \cup P_3 \cup P_4 - P_1 \cup P_2 \cup P_3 \cup P_4 =$$

$$= \{1,3\} + \{1,4\} + \{2,3\} + \{2,4\} - \{1,3,4\} - \{1,2,3\} - \{1,2,3,4\}$$
$$- \{1,2,3,4\} - \{1,2,4\} - \{2,3,4\} + \{1,2,3,4\} + \{1,2,3,4\}$$
$$+ \{1,2,3,4\} + \{1,2,3,4\} - \{1,2,3,4\}$$

- In the general case

$$\bar{F}_T(t) = \bar{F}_{\{1,3\}}(t) + \bar{F}_{\{1,4\}}(t) + \bar{F}_{\{2,3\}}(t) + \bar{F}_{\{2,4\}}(t) - \bar{F}_{\{1,2,3\}}(t) - \bar{F}_{\{1,3,4\}}(t) - \bar{F}_{\{1,2,4\}}(t) - \bar{F}_{\{2,3,4\}}(t) + \bar{F}_{\{1,2,3,4\}}(t)$$

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- In the general case

$$\bar{F}_T(t) = \bar{Q}(\bar{F}_1(t), \bar{F}_2(t), \bar{F}_3(t), \bar{F}_4(t)),$$

with $\bar{Q}(u_1, u_2, u_3, u_4) = \hat{C}(u_1, 1, u_3, 1) + \hat{C}(u_1, 1, 1, u_4) + \hat{C}(1, u_2, u_3, 1) + \hat{C}(1, u_2, 1, u_4) - \hat{C}(u_1, u_2, u_3, 1) - \hat{C}(u_1, 1, u_3, u_4) - \hat{C}(u_1, u_2, 1, u_4) - \hat{C}(1, u_2, u_3, u_4) + \hat{C}(u_1, u_2, u_3, u_4)$

- In the general case

$\bar{F}_T(t) = \bar{F}_{\{1,3\}}(t) + \bar{F}_{\{1,4\}}(t) + \bar{F}_{\{2,3\}}(t) + \bar{F}_{\{2,4\}}(t) - \bar{F}_{\{1,2,3\}}(t) - \bar{F}_{\{1,3,4\}}(t) - \bar{F}_{\{1,2,4\}}(t) - \bar{F}_{\{2,3,4\}}(t) + \bar{F}_{\{1,2,3,4\}}(t)$

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$$\bar{F}_T(t) = \bar{Q}(\bar{F}_1(t), \bar{F}_2(t), \bar{F}_3(t), \bar{F}_4(t)),$$

with $\bar{Q}(u_1, u_2, u_3, u_4) = \hat{C}(u_1, 1, u_3, 1) + \hat{C}(u_1, 1, 1, u_4) + \hat{C}(1, u_2, u_3, 1) + \hat{C}(1, u_2, 1, u_4) - \hat{C}(u_1, u_2, u_3, 1) - \hat{C}(u_1, 1, u_3, u_4) - \hat{C}(u_1, u_2, 1, u_4) - \hat{C}(1, u_2, u_3, u_4) + \hat{C}(u_1, u_2, u_3, u_4)$

- IND case

$$\bar{F}_T(t) = \bar{Q}(\bar{F}_1(t), \bar{F}_2(t), \bar{F}_3(t), \bar{F}_4(t)),$$

with $\bar{Q}(u_1, u_2, u_3, u_4) = u_1 u_3 + u_1 u_4 + u_2 u_3 + u_2 u_4 - u_1 u_2 u_3 - u_1 u_3 u_4 - u_1 u_2 u_4 - u_2 u_3 u_4 + u_1 u_2 u_3 u_4$

- ID case

$$\bar{F}_T(t) = \bar{q}(\bar{F}(t)),$$

with $\bar{q}(u) = \hat{C}(u,1,u,1) + \hat{C}(u,1,1,u) + \hat{C}(1,u,u,1) + \hat{C}(1,u,1,u) - \hat{C}(u,u,u,1) - \hat{C}(u,1,u,u) - \hat{C}(u,u,1,u) - \hat{C}(1,u,u,u) + \hat{C}(u,u,u,u)$

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- IID case

$$\bar{F}_T(t) = \bar{q}(\bar{F}(t)),$$

with $\bar{q}(u) = 4u^2 - 4u^3 + u^4$

Exercise 7. Compute the minimal and maximal signatures of a system with four components.

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$$a = \{0, 4, -4, 1\}$$

Exercise 7. Compute the minimal and maximal signatures of a system with four components.

Dual system

$T^* = max(min(X_1, X_2), min(X_3, X_4))$

- $T = min(max(X_1, X_2), max(X_3, X_4))$
- IID case

$$\bar{F}_T(t) = \bar{q}(\bar{F}(t)),$$

with $\bar{q}(u) = 4u^2 - 4u^3 + u^4$

- Minimal signature $a = (0, 4, -4, 1)$

Exercise 7. Compute the minimal and maximal signatures of a system with four components.

- $T = min(max(X_1, X_2), max(X_3, X_4))$
- IID case

$$\bar{F}_T(t) = \bar{q}(\bar{F}(t)),$$

with $\bar{q}(u) = 4u^2 - 4u^3 + u^4$

- Minimal signature $a = (0, 4, -4, 1)$
- Maximal signature is the minimal signature of dual sistem $T^* = max(min(X_1, X_2), min(X_3, X_4))$ (cfr. Exercise 5), that is $b = (0, 2, 0 - 1)$.

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- We check, for example the arrow $16 \to 6$, for ST, HR and LR orders
- $T_{16} = max(min(X_1, X_2), min(X_3, X_4))$ and $T_6 = X_{2:3}$

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- We check, for example the arrow $16 \to 6$, for ST, HR and LR orders
- $T_{16} = max(min(X_1, X_2), min(X_3, X_4))$ and $T_6 = X_{2:3}$
- IID case (cfr Ex.5 and the first lesson) $\bar{Q}_6(u,u,u,u)$

$\bar{Q}_{16}(u,u,u,u) = \bar{q}_{16}(u) = 2u^2 - u^4$ and $\bar{q}_6(u) = \bar{q}_{2:3}(u) = 3u^2 - 2u^3$

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- We check, for example the arrow $16 \to 6$, for ST, HR and LR orders
- $T_{16} = max(min(X_1, X_2), min(X_3, X_4))$ and $T_6 = X_{2:3}$
- IID case (cfr Ex.5 and the first lesson)

$$\bar{q}_{16}(u) = 2u^2 - u^4 \quad \text{and} \quad \bar{q}_6(u) = \bar{q}_{2:3}(u) = 3u^2 - 2u^3$$

- $T_{16} \leq_{ST} T_6 \iff \bar{q}_{16}(u) \leq \bar{q}_6(u)$ in $(0,1)$

$$2u^2 - u^4 \leq 3u^2 - 2u^3 \quad \text{in} \quad (0,1)$$

$$2 - u^2 \leq 3 - 2u \iff 1 \geq 2u - u^2 \iff 1 \geq u(2-u)$$

$u \in (0,1)$, $2 - u \geq 1$

$\forall u \in (0,1)$

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- $T_{16} = max(min(X_1, X_2), min(X_3, X_4))$ and $T_6 = X_{2:3}$
- IID case (cfr Ex.5 and the first lesson)
$$\bar{q}_{16}(u) = 2u^2 - u^4 \quad \text{and} \quad \bar{q}_6(u) = \bar{q}_{2:3}(u) = 3u^2 - 2u^3$$
- $T_{16} \leq_{ST} T_6 \iff \bar{q}_{16}(u) \leq \bar{q}_6(u)$ in $(0,1)$
$$2u^2 - u^4 \leq 3u^2 - 2u^3 \quad \text{in} \quad (0,1)$$
- $T_{16} \leq_{HR} T_6 \iff \bar{q}_6(u)/\bar{q}_{16}(u)$ decreases in $(0,1)$
$$\frac{\bar{q}_6(u)}{\bar{q}_{16}(u)} = \frac{3-2u}{2-u^2} \text{ decreases in } (0,1)$$

$\bar{q}_{16}(u) = 2u^2 - u^4 \Rightarrow \bar{q}'_{16}(u) = 4u - 4u^3$

$\bar{q}_6(u) = 3u^2 - 2u^3 \Rightarrow \bar{q}'_6(u) = 6u - 6u^2$

- $T_{16} \leq_{LR} T_6 \iff \bar{q}'_6(u)/\bar{q}'_{16}(u)$ decreases in $(0,1)$

$$\frac{\bar{q}'_6(u)}{\bar{q}'_{16}(u)} = \frac{6-6u}{4-4u^2} \text{ decreases in } (0,1)$$

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- We check, for example the no arrow 16 $\not\to$ 5, for ST, HR and LR orders
- $T_{16} = max(min(X_1, X_2), min(X_3, X_4))$ and $T_5 = min(X_1, max(X_2, X_3))$

minimal signature

$a = (0, 2, -1)$

$\overline{q}_5(u) = 0 \cdot u + 2u^2 - 1 \cdot u^3$

Exercise 9. Check a no arrow in the figures for the ST, HR and LR orders of systems with IID components.

- We check, for example the no arrow $16 \not\to 5$, for ST, HR and LR orders
- $T_{16} = max(min(X_1, X_2), min(X_3, X_4))$ and $T_5 = min(X_1, max(X_2, X_3))$
- IID case

$$\bar{q}_{16}(u) = 2u^2 - u^4 \quad \text{and} \quad \bar{q}_5(u) = 2u^2 - u^3$$

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- $T_{16} = max(min(X_1, X_2), min(X_3, X_4))$ and $T_5 = min(X_1, max(X_2, X_3))$
- IID case

$$\bar{q}_{16}(u) = 2u^2 - u^4 \quad \text{and} \quad \bar{q}_5(u) = 2u^2 - u^3$$

- $T_{16} \not\leq_{ST} T_5$ because

$$2u^2 - u^4 \geq 2u^2 - u^3 \quad \text{in} \quad (0,1) \qquad \bar{q}_{16}(u) \geq \bar{q}_6(u)$$

$$\Leftrightarrow 2 - u^2 \geq 2 - u \Leftrightarrow u^2 \leq u, \ \forall u \in (0,1)$$

Exercise 9. Check a no arrow in the figures for the ST, HR and LR orders of systems with IID components.

- We check, for example the no arrow $16 \not\to 5$, for ST, HR and LR orders
- $T_{16} = max(min(X_1, X_2), min(X_3, X_4))$ and $T_5 = min(X_1, max(X_2, X_3))$
- IID case

$$\bar{q}_{16}(u) = 2u^2 - u^4 \quad \text{and} \quad \bar{q}_5(u) = 2u^2 - u^3$$

- $T_{16} \not\leq_{ST} T_5$ because

$$2u^2 - u^4 \geq 2u^2 - u^3 \quad \text{in} \quad (0,1)$$

- $T_{16} \not\leq_{HR} T_5$ because

$$\frac{\bar{q}_5(u)}{\bar{q}_{16}(u)} = \frac{2-u}{2-u^2} \text{ is not monotone in } (0,1)$$

$\bar{q}'_{16}(u) = 4u - 4u^3$

$\bar{q}'_5(u) = 4u - 3u^2$

- $T_{16} \not\leq_{LR} T_6$ because

$$\frac{\bar{q}'_5(u)}{\bar{q}'_{16}(u)} = \frac{4-3u}{4-4u^3} \text{ is not monotone in } (0,1)$$

Exercise 10. Check if $X_i \leq_{HR} X_{2:2}$ holds for IND components.

$i = 1, 2$ $\quad\quad$ $i = 1, 2$

- $X_i \leq_{HR} X_{2:2} \iff \bar{Q}_{2:2}/\bar{Q}_i$ is decreasing in $(0,1)^2$

Exercise 10. Check if $X_i \leq_{HR} X_{2:2}$ holds for IND components.

- $X_i \leq_{HR} X_{2:2} \iff \bar{Q}_{2:2}/\bar{Q}_i$ is decreasing in $(0,1)^2$
- IND case

$$\bar{Q}_{2:2}(u_1, u_2) = u_1 + u_2 - u_1 u_2 \quad \text{and} \quad \bar{Q}_i(u_1, u_2) = u_i \qquad i = 1, 2$$

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- $\frac{\bar{Q}_{2:2}}{\bar{Q}_2} = \frac{u_1 + u_2 - u_1 u_2}{u_2}$ is increasing in u_1 and decreasing in u_2.
- $X_i \leq_{HR} X_{2:2}$ NOT hold for IND components.

Exercise 11. Check if $X_i \leq_{HR} X_{2:2}$ holds for IND HR-ordered components.

- $F_1 \geq_{HR} F_2$

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- $F_1 \geq_{HR} F_2$
- $X_i \leq_{HR} X_{2:2} \iff \bar{H}(v_1, v_2) = \frac{\bar{Q}_{2:2}(v_1, v_1 v_2)}{\bar{Q}_i(v_1, v_1 v_2)}$ is decreasing in $(0,1)^2$ $\quad i = 1, 2$

$\bar{Q}_{2:2}(v_1, v_1 v_2) = v_1 + v_1 v_2 - v_1^2 v_2$

Exercise 11. Check if $X_i \leq_{HR} X_{2:2}$ holds for IND HR-ordered components.

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- IND case

$\bar{Q}_{2:2}(v_1, v_1 v_2) = v_1 + v_1 v_2 - v_1^2 v_2$ and $\bar{Q}_1(v_1, v_1 v_2) = v_1, \bar{Q}_2(v_1, v_1 v_2) = v_1 v_2$

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$\bar{Q}_{2:2}(v_1, v_1 v_2) = v_1 + v_1 v_2 - v_1^2 v_2$ and $\bar{Q}_1(v_1, v_1 v_2) = v_1$, $\bar{Q}_2(v_1, v_1 v_2) = v_1 v_2$

- $\frac{\bar{Q}_{2:2}(v_1, v_1 v_2)}{\bar{Q}_1(v_1, v_1 v_2)} = 1 + v_2 - v_1 v_2$ $= \frac{v_1(1 + v_2 - v_1 v_2)}{v_1}$

is decreasing in v_1 and is increasing in v_2 in $(0,1)^2$, so $X_1 \leq_{HR} X_{2:2}$ NOT holds for IND HR-ordered components.

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- $\frac{\bar{Q}_{2:2}(v_1, v_1 v_2)}{\bar{Q}_2(v_1, v_1 v_2)} = \frac{1 + v_2 - v_1 v_2}{v_2} = \frac{v_1(1 + v_2 - v_1 v_2)}{v_1 v_2}$

is decreasing in $(0,1)^2$ so $X_2 \leq_{HR} X_{2:2}$ holds for IND HR-ordered components.

Exercise 12. Check if $X_i \leq_{HR} X_{2:2}$ holds for dependent components with the Clayton copula in the slides.

- $X_i \leq_{HR} X_{2:2} \iff \bar{Q}_{2:2}/\bar{Q}_i$ is decreasing in $(0,1)^2$

Clayton copula $\hat{C}(u_1, u_2) = \dfrac{u_1 u_2}{u_1 + u_2 - u_1 u_2}$

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- Dependent case with Clayton copula $i = 1, 2$

$\bar{Q}_{2:2}(u_1, u_2) = u_1 + u_2 - \hat{C}(u_1, u_2)$ and $\bar{Q}_i(u_1, u_2) = u_i$, that is

$$\bar{Q}_{2:2}(u_1, u_2) = u_1 + u_2 - \frac{u_1 u_2}{u_1 + u_2 - u_1 u_2}.$$

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- $\frac{\bar{Q}_{2:2}}{\bar{Q}_1} = \frac{u_1^2 + u_2^2 - u_1^2 u_2 - u_1 u_2^2 + u_1 u_2}{u_1(u_1 + u_2 - u_1 u_2)}$ is decreasing in u_1 and increasing in u_2.

- $\frac{\bar{Q}_{2:2}}{\bar{Q}_2} = \frac{u_1^2 + u_2^2 - u_1^2 u_2 - u_1 u_2^2 + u_1 u_2}{u_2(u_1 + u_2 - u_1 u_2)}$ is increasing in u_1 and decreasing in u_2.

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- $\frac{\bar{Q}_{2:2}}{\bar{Q}_1} = \frac{u_1^2+u_2^2-u_1^2u_2-u_1u_2^2+u_1u_2}{u_1(u_1+u_2-u_1u_2)}$ is decreasing in u_1 and increasing in u_2.

- $\frac{\bar{Q}_{2:2}}{\bar{Q}_2} = \frac{u_1^2+u_2^2-u_1^2u_2-u_1u_2^2+u_1u_2}{u_2(u_1+u_2-u_1u_2)}$ is increasing in u_1 and decreasing in u_2.

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$\bar{Q}_{2:2}(u_1, u_2) = u_1 + u_2 - \hat{C}(u_1, u_2)$ and $\bar{Q}_i(u_1, u_2) = u_i$, that is

$$\bar{Q}_{2:2}(u_1, u_2) = u_1 + u_2 - \frac{u_1 u_2}{u_1 + u_2 - u_1 u_2}.$$

- $\frac{\bar{Q}_{2:2}}{\bar{Q}_1} = \frac{u_1^2 + u_2^2 - u_1^2 u_2 - u_1 u_2^2 + u_1 u_2}{u_1(u_1 + u_2 - u_1 u_2)}$ is decreasing in u_1 and increasing in u_2.
- $\frac{\bar{Q}_{2:2}}{\bar{Q}_2} = \frac{u_1^2 + u_2^2 - u_1^2 u_2 - u_1 u_2^2 + u_1 u_2}{u_2(u_1 + u_2 - u_1 u_2)}$ is increasing in u_1 and decreasing in u_2.
- $X_i \leq_{HR} X_{2:2}$ NOT hold for dependent components with the Clayton copula.

Exercise 13. Check an arrow in the tables and figure for the ST and HR orders of systems with IND components.

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- We check, for example the arrow $5 \to 9$, for ST order.

ST	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
2	2	1	1	2	2	1	2	2	1	2	2	2	2	2	2	2	2
3	0	2	1	2	1	2	2	1	2	2	2	2	2	2	2	2	2
4	0	0	2	0	2	2	0	2	2	2	2	2	2	2	2	2	2
5	0	0	0	2	1	1	2	1	1	2	2	2	2	2	2	2	2
6	0	0	0	0	2	1	0	2	1	2	2	2	2	2	2	2	2
7	0	0	0	0	0	2	0	0	2	2	2	2	2	2	2	2	2

Exercise 13. Check an arrow in the tables and figure for the ST and HR orders of systems with IND components.

- We check, for example the arrow $5 \to 9$, for ST order.

ST	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
2	2	1	1	2	2	1	2	2	1	2	2	2	2	2	2	2	2
3	0	2	1	2	1	2	2	1	2	2	2	2	2	2	2	2	2
4	0	0	2	0	2	2	0	2	2	2	2	2	2	2	2	2	2
5	0	0	0	2	1	1	2	1	1	2	2	2	2	2	2	2	2
6	0	0	0	0	2	1	0	2	1	2	2	2	2	2	2	2	2
7	0	0	0	0	0	2	0	0	2	2	2	2	2	2	2	2	2

- The value 1 indicates that $T_5 \leq_{ST} T_j$ holds for all ST ordered components (i denotes the row and j the column).

- $T_5 = min(X_3, max(X_1, X_2))$ and $T_9 = X_2$

- $T_5 = min(X_3, max(X_1, X_2))$ and $T_9 = X_2$
- IND case

  $\bar{Q}_5(u_1, u_2, u_3) = u_1 u_3 + u_2 u_3 - u_1 u_2 u_3$ and $\bar{Q}_9(u_1, u_2, u_3) = u_2$

- $T_5 = min(X_3, max(X_1, X_2))$ and $T_9 = X_2$
- IND case

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- $\bar{Q}_5 \leq \bar{Q}_9$ if $u_1 \geq u_2 \geq u_3$, so

- $T_5 = min(X_3, max(X_1, X_2))$ and $T_9 = X_2$
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- $\bar{Q}_5 \leq \bar{Q}_9$ if $u_1 \geq u_2 \geq u_3$, so

$$T_5 \leq_{ST} T_9 \text{ for ST ordered component.}$$

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HR	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
2	2	1	1	1	1	1	2	2	1	1	1	1	1	2	1
3	0	2	1	0	0	1	2	1	2	0	1	1	1	1	2
4	0	0	2	0	0	0	0	2	2	0	0	0	0	0	0
5	0	0	0	2	0	0	2	1	1	0	0	1	1	1	1
6	0	0	0	0	2	0	0	2	1	0	0	0	1	0	2
7	0	0	0	0	0	2	0	0	2	0	0	0	1	2	1

- We check, for example the arrow $1 \to 9$, for HR order.

HR	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
2	2	1	1	1	1	1	2	2	1	1	1	1	1	2	1
3	0	2	1	0	0	1	2	1	2	0	1	1	1	1	2
4	0	0	2	0	0	0	0	2	2	0	0	0	0	0	0
5	0	0	0	2	0	0	2	1	1	0	0	1	1	1	1
6	0	0	0	0	2	0	0	2	1	0	0	0	1	0	2
7	0	0	0	0	0	2	0	0	2	0	0	0	1	2	1

- The value 2 indicates that $T_i \leq_{HR} T_j$ holds for any F_1, F_2, F_3 (i denotes the row and j the column).

- $T_1 = X_{1:3} = min(X_1, X_2, X_3)$ and $T_9 = X_2$

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$$\bar{Q}_1(u_1, u_2, u_3) = u_1 u_2 u_3 \quad \text{and} \quad \bar{Q}_9(u_1, u_2, u_3) = u_2$$

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$$\frac{\bar{Q}_9}{\bar{Q}_1} = \frac{1}{u_1 u_3} \text{ is decreasing in } (0,1)^3.$$

Exercise 14. Check if the IFR class is preserved in a system with four IID components.

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- $T = max(min(x_1, x_2), min(x_3, x_4))$
- IID case (Cfr. Ex. 5)

$$\bar{q}(u) = 2u^2 - u^4$$

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- $T = max(min(x_1, x_2), min(x_3, x_4))$
- IID case (Cfr. Ex. 5)

$$\bar{q}(u) = 2u^2 - u^4$$

- Compute

$$\alpha(u) = u\frac{\bar{q}'(u)}{\bar{q}(u)} = \frac{4 - 4u^2}{2 - u^2}$$

Exercise 14. Check if the IFR class is preserved in a system with four IID components.

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$$\bar{q}(u) = 2u^2 - u^4$$

- Compute

$$\alpha(u) = u\frac{\bar{q}'(u)}{\bar{q}(u)} = \frac{4 - 4u^2}{2 - u^2}$$

- $\alpha(u)$ decreases for $u \in (0, 1) \rightarrow$ IFR class is preserved in system T.

$\Rightarrow$ NBU class is preserved ; $\Rightarrow$ IFRA class is preserved