Linear stochastic differential equation. Cox-Ingersoll-Ross equation

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Lecture 3

Linear stochastic differential equation. Cox-Ingersoll-Ross equation

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Simple Itô formula for fBm with $H > 1/2$

Let $H > 1/2$, $F : R \to R$ is continuously differentiable. Then, taking into account that a quadratic variation of fBm equals zero,

$$F(B_t^H) = F(0) + \int_0^t F'(B_s^H) dB_s^H.$$

Linear stochastic differential equation involving fractional Brownian motion

Consider an equation

$$dZ_t = aZ_t dt + \sigma Z_t dB_t^H, \quad t \geq 0, \quad a \in \mathbb{R}, \sigma > 0, H > 1/2. \tag{1}$$

Its solution exists, is unique and has an explicit form

$$Z_t = Z_0 \exp\{at + \sigma B_t^H\}, \quad t \geq 0.$$

Cox-Ingersoll-Ross process with standard Brownian motion

It is the unique solution of standard SDE

$$dr_t = (a - br_t)dt + \sigma\sqrt{r_t}dW_t, t \geq 0.$$

Existence-uniqueness follows from results of Yamada-Kunita-Watanabe: if the drift and diffusion coefficients are of linear growth, drift coefficient is Lipschitz and diffusion coefficient is Hölder of order $1/2$ or bigger, then the solution exists and is unique.
It is a nonnegative stochastic process, and if $2a \geq \sigma^2$, it is strictly positive (again Kunita-Watanabe concerning nonnegativity, and Gikhman, Skorokhod concerning positivity.) Why to consider CIR? Two reasons:
1) To get the process for which the diffusion coefficient is proportional to the process itself, but not with power 1, a bit less.
2) To get the non-negative and even positive process in order to model interest rate, for example.
Since Ornstein-Uhlenbeck process is Gaussian, it can not serve for the 2nd goal (the 1st also not).

Fractional CIR process with $k = 0$ and $H > 2/3$

Potentially, our goal is to consider the stochastic differential equation of the following form:

$$dX_t = (k + aX_t)dt + \sigma\sqrt{X_t}dB_t^H, \quad t \geq 0, \quad a, k \in \mathbb{R}, \quad X_0, \sigma > 0. \quad (2)$$

To start with, consider the particular case of stochastic differential equation (2) where $k = 0$:

$$dX_t = \tilde{a}X_t dt + \tilde{\sigma}\sqrt{X_t}dB_t^H, \quad t \geq 0, \quad \tilde{a} \in \mathbb{R}, \quad X_0, \tilde{\sigma} > 0, \quad (3)$$

$B^H = \{B^H, t \geq 0\}$ is a fractional Brownian motion with $H > 2/3$.

Existence and uniqueness of the solution in case $H \in (2/3, 1)$

Consider the integral $\int_0^T \sigma\sqrt{X_t}dB_t^H$. If $H > 2/3$ and X is non-negative, then $\sqrt{X_t}$ is Hölder up to $1/3$, $1/3 + 2/3 > 1$, and the integral exists as the Riemann-Stieltjes integral.

If $H > 2/3$, the equation (3) has a unique solution until the first moment of reaching zero, and the integral $\int_0^t \sqrt{X_s}dB_s^H$ exists as a pathwise Riemann-Stieltjes sums limit.

Connection with the fractional Ornstein-Uhlenbeck process

Denote $\tau_0 := \inf\{t > 0 : X_t = 0\}$ and consider the trajectories of the process $\{X_t, t \geq 0\}$ on $[0, \tau_0)$. After substitution $Y_t = \sqrt{X_t}$ and using the Itô formula for integrals with respect to fractional Brownian motion, we obtain:

$$dY_t = \frac{dX_t}{2\sqrt{X_t}} = \frac{\tilde{a} X_t dt}{2\sqrt{X_t}} + \frac{\tilde{\sigma}}{2} dB_t^H. \tag{4}$$

Denoting $a = \tilde{a}/2$, $\sigma = \tilde{\sigma}/2$, we get

$$dY_t = aY_t dt + \sigma dB_t^H \tag{5}$$

with the initial condition $Y_0 = \sqrt{X_0}$.
So, in the case of $H > 2/3$, the solution $\{X_t, t \in [0, \tau_0)\}$ of the equation (3) is the square of the fractional Ornstein–Uhlenbeck process until it reaches zero.

Definition of the fractional Cox–Ingersoll–Ross process for an arbitrary $H \in (0,1)$ and zero "mean" parameter

Let $H \in (0,1)$ be an arbitrary Hurst index, $\{Y_t, t \geq 0\}$ be a fractional Ornstein-Uhlenbeck process, and τ be the first moment of reaching zero by the latter.

Definition 1.1

The fractional Cox-Ingersoll-Ross process (with zero "mean" parameter) is the process $\{X_t, t \geq 0\}$ such that for all $t \geq 0$, $\omega \in \Omega$:

$$X_t(\omega) = Y_t^2(\omega) 1_{\{t < \tau(\omega)\}}. \tag{6}$$

The definition of the fractional CIR process is "natural"

Let us show that such definition is natural, i.e. the fractional CIR process satisfies the equation similar (in some way) to the "standard" equation (3).

Stratonovich integral

Definition 1.2

Let $\{X_t, t \geq 0\}$, $\{Y_t, t \geq 0\}$ be random processes.
The pathwise Stratonovich integral $\int_0^T X_s \circ dY_s$ is a pathwise limit of the following sums:

$$\sum_{i=1}^{n} \frac{X_{t_k} + X_{t_{k-1}}}{2}(Y_{t_k} - Y_{t_{k-1}}), \tag{7}$$

as the mesh of the partition $0 = t_0 < t_1 < ... < t_n = T$ tends to zero, in case if this limit exists.

Equation for the fractional CIR process

Theorem 1.3

Let τ be the first moment of zero hitting by the fractional Ornstein-Uhlenbeck process with parameters $a \in \mathbb{R}$, $\sigma > 0$ and $H > 1/2$.

Then, for $0 \leq t \leq \tau$, the corresponding fractional CIR process satisfies the following SDE:

$$dX_t = 2aX_t dt + 2\sigma\sqrt{X_t} \circ dB_t^H, \tag{8}$$

where $X_0 = Y_0^2 > 0$ and the integral with respect to the fractional Brownian motion is defined as the pathwise Stratonovich integral.

Zero hitting time finiteness

The next natural question regarding the fractional CIR process is finiteness of its zero hitting time moment.

It is obvious that it coincides with the respective moment of the corresponding fractional Ornstein-Uhlenbeck process $\{Y_t, t \geq 0\}$.

Preliminary remarks

Now, let $\{Y_t, t \geq 0\}$ be a fractional Ornstein-Uhlenbeck process. Recall that it is the solution of the SDE

$$dY_t = aY_t dt + \sigma dB_t^H, \quad t \geq 0, \quad a \in \mathbb{R}, \sigma > 0. \tag{9}$$

Let τ be the first moment of reaching zero by the latter.

Preliminary remarks

Recall again that Y can be written explicitly as

$$Y_t = e^{at}\left(Y_0 + \sigma \int_0^t e^{-as} dB_s^H\right). \tag{10}$$

Preliminary remarks

From the formula (10), we see that the first zero hitting moment of the process Y coincides with the first time the integral J_t reaches the level $-Y_0/\sigma$.
This integral is a normally distributed random variable with zero mean. Therefore, due to the symmetry of the normal distribution, the probability that the integral J_t hits the negative level $-Y_0/\sigma$ coincides with the probability of reaching the positive level Y_0/σ.
Thus, the problem of studying the probability that integral J_t hits the level $x > 0$ in a finite time arises.

Main theorem

Let τ be the first moment of zero hitting by the fractional Ornstein-Uhlenbeck process (and consequently by the corresponding fractional CIR process with zero "mean" parameter).

Theorem 1.4

(1) *If* $a \le 0$, *then* $\mathbb{P}(\tau < \infty) = 1$.

(2) *If* $a > 0$, *then* $\mathbb{P}(\tau < \infty) \in (0, 1)$, *and we have the upper bound*

$$\mathbb{P}(\tau < \infty) \le C_1 \left(\frac{Y_0}{\sigma}\right)^{\frac{1}{H}-2} \exp\left(-\frac{a^{2H} Y_0^2}{\sigma^2 \Gamma(2H+1)}\right), \qquad (11)$$

where $C_1 > 0$ *is a constant.*

Definition of the fractional CIR process

Consider the process $Y = \{Y_t, t \geq 0\}$ that satisfies the following SDE until its first zero hitting:

$$dY_t = \frac{1}{2}\left(\frac{k}{Y_t} - aY_t\right)dt + \frac{\sigma}{2}dB_t^H, \quad Y_0 > 0, \tag{12}$$

where $a, k \in \mathbb{R}$, $\sigma > 0$ and $\{B_t^H, t \geq 0\}$ is a fractional Brownian motion with the Hurst parameter $H \in (0, 1)$.

Definition of the fractional CIR process

Definition 1.5

Let $H \in (0,1)$ be an arbitrary Hurst index, $\{Y_t, t \geq 0\}$ be the process that satisfies the equation (12) and τ be the first moment of reaching zero by the latter.
The fractional Cox-Ingersoll-Ross process is the process $\{X_t, t \geq 0\}$ such that for all $t \geq 0, \omega \in \Omega$:

$$X_t(\omega) = Y_t^2(\omega) 1_{\{t < \tau(\omega)\}}. \tag{13}$$

Definition of the fractional CIR process

Remark 1.6

If $k = 0$, then Y is a fractional Ornstein-Uhlenbeck process and this definition coincides with the definition of the fractional CIR process with zero "mean" parameter given before.

Equation for the fractional CIR process

Similarly to the case $k = 0$, the definition of the fractional CIR process is natural as the following theorem holds:

Theorem 1.7

Let τ be the first moment of hitting zero by Y.
For $0 \leq t \leq \tau$ the fractional CIR process satisfies the following SDE:

$$dX_t = (k - aX_t)dt + \sigma\sqrt{X_t} \circ dB_t^H, \tag{14}$$

where $X_0 = Y_0^2 > 0$ and the integral with respect to the fractional Brownian motion is defined as the pathwise Stratonovich integral.

Proof.

Denote $\tau = \inf\{s > 0 : Y_s = 0\}$ and for a fixed $\omega \in \Omega$ consider an arbitrary $t = t(\omega) < \tau$.
According to Definition 1.5,

$$X_t = Y_t^2 = \left(\sqrt{X_0} + \frac{1}{2} \int_0^t \left(\frac{k}{Y_s} - aY_s \right) ds + \frac{\sigma}{2} B_t^H \right)^2 . \qquad (15)$$

Consider an arbitrary partition of the interval $[0, t]$:

$$0 = t_0 < t_1 < t_2 < \ldots < t_{n-1} < t_n = t.$$

Using (15), we get

$$
\begin{aligned}
X_t &= \sum_{i=1}^{n} \left(X_{t_i} - X_{t_{i-1}}\right) + X_0 = \\
&= \sum_{i=1}^{n} \left(\left[\sqrt{X_0} + \frac{1}{2} \int_0^{t_i} \left(\frac{k}{Y_s} - aY_s\right) ds + \frac{\sigma}{2} B_{t_i}^H \right]^2 - \right. \\
&\quad \left. - \left[\sqrt{X_0} + \frac{1}{2} \int_0^{t_{i-1}} \left(\frac{k}{Y_s} - aY_s \right) ds + \frac{\sigma}{2} B_{t_{i-1}}^H \right]^2 \right) + X_0 = \\
&= X_0 + \sum_{i=1}^{n} \left[2\sqrt{X_0} + \frac{1}{2} \left(\int_0^{t_i} \left(\frac{k}{Y_s} - aY_s \right) ds + \int_0^{t_{i-1}} \left(\frac{k}{Y_s} - aY_s \right) ds \right) \right. \\
&\quad \left. + \frac{\sigma}{2} \left(B_{t_i}^H + B_{t_{i-1}}^H \right) \right] \left[\frac{1}{2} \int_{t_{i-1}}^{t_i} \left(\frac{k}{Y_s} - aY_s \right) ds + \frac{\sigma}{2} \left(B_{t_i}^H - B_{t_{i-1}}^H \right) \right].
\end{aligned}
$$

Expanding the brackets in the last expression, we obtain:

$$X_t = \sum_{i=1}^{n} \sqrt{X_0} \int_{t_{i-1}}^{t_i} \left(\frac{k}{Y_s} - aY_s \right) ds +$$
$$+ \frac{1}{4} \sum_{i=1}^{n} \left(\int_0^{t_i} \left(\frac{k}{Y_s} - aY_s \right) ds + \int_0^{t_{i-1}} \left(\frac{k}{Y_s} - aY_s \right) ds \right) \times$$
$$\times \int_{t_{i-1}}^{t_i} \left(\frac{k}{Y_s} - aY_s \right) ds + \frac{\sigma}{4} \sum_{i=1}^{n} \left(B_{t_i}^H + B_{t_{i-1}}^H \right) \int_{t_{i-1}}^{t_i} \left(\frac{k}{Y_s} - aY_s \right) ds +$$
$$+ \sigma \sqrt{X_0} \sum_{i=1}^{n} \left(B_{t_i}^H - B_{t_{i-1}}^H \right) + \frac{\sigma^2}{4} \sum_{i=1}^{n} \left(B_{t_i}^H - B_{t_{i-1}}^H \right) \left(B_{t_i}^H + B_{t_{i-1}}^H \right) +$$
$$+ \frac{\sigma}{4} \sum_{i=1}^{n} \left(\int_0^{t_i} \left(\frac{k}{Y_s} - aY_s \right) ds + \int_0^{t_{i-1}} \left(\frac{k}{Y_s} - aY_s \right) ds \right) \left(B_{t_i}^H - B_{t_{i-1}}^H \right). \tag{16}$$

Let the mesh Δt of the partition tend to zero. The first three summands

$$\sum_{i=1}^{n} \sqrt{X_0} \int_{t_{i-1}}^{t_i} \left(\frac{k}{Y_s} - aY_s \right) ds +$$
$$+ \frac{1}{4} \sum_{i=1}^{n} \left(\int_0^{t_i} \left(\frac{k}{Y_s} - aY_s \right) ds + \int_0^{t_{i-1}} \left(\frac{k}{Y_s} - aY_s \right) ds \right) \times$$
$$\times \int_{t_{i-1}}^{t_i} \left(\frac{k}{Y_s} - aY_s \right) ds + \frac{\sigma}{4} \sum_{i=1}^{n} \left(B_{t_i}^H + B_{t_{i-1}}^H \right) \int_{t_{i-1}}^{t_i} \left(\frac{k}{Y_s} - aY_s \right) ds \to$$
$$\to \int_0^t \left(\frac{k}{Y_s} - aY_s \right) \left(\sqrt{X_0} + \frac{1}{2} \int_0^s \left(\frac{k}{Y_u} - aY_u \right) du + \frac{\sigma}{2} B_s^H \right) ds =$$
$$= \int_0^t \left(k - aY_s^2 \right) ds = \int_0^t \left(k - aX_s \right) ds, \quad \Delta t \to 0, \tag{17}$$

and the last three summands

$$\sigma\sqrt{X_0}\sum_{i=1}^{n}\left(B_{t_i}^H - B_{t_{i-1}}^H\right) + \frac{\sigma^2}{4}\sum_{i=1}^{n}\left(B_{t_i}^H - B_{t_{i-1}}^H\right)\left(B_{t_i}^H + B_{t_{i-1}}^H\right) +$$
$$+\frac{\sigma}{4}\sum_{i=1}^{n}\left(\int_0^{t_i}\left(\frac{k}{Y_s} - aY_s\right)ds + \int_0^{t_{i-1}}\left(\frac{k}{Y_s} - aY_s\right)ds\right)\left(B_{t_i}^H - B_{t_{i-1}}^H\right) \to$$
$$\to \sigma\int_0^t\left(\sqrt{X_0} + \frac{1}{2}\int_0^s\left(\frac{k}{Y_u} - aY_u\right)du + \frac{\sigma}{2}B_s^H\right)\circ dB_s^H =$$
$$= \sigma\int_0^t Y_s \circ dB_s^H = \sigma\int_0^t \sqrt{X_s}\circ dB_s^H, \quad \Delta t \to 0. \tag{18}$$

Thus, the fractional Cox–Ingersoll–Ross process, introduced in Definition 1.5, satisfies the SDE of the form

$$X_t = X_0 + \int_0^t (k - aX_s)\, ds + \sigma \int_0^t \sqrt{X_s} \circ dB_s^H, \tag{19}$$

where $\int_0^t \sqrt{X_s} \circ dB_s^H$ is a pathwise Stratonovich integral.

Remark 1.8

The passage to the limit in (18) is correct since the left-hand side of the equation (16) does not depend on the partition and the limit in formula (17) exists as the pathwise Riemann integral, therefore the corresponding pathwise Stratonovich integral also exists.

Hitting zero by the fractional CIR process (case $k > 0$)

Just as in the case $k = 0$, let us consider the question of finiteness of the zero hitting time moment by the fractional CIR process.

Theorem 1.9

Let $\{B_t^H, t \geq 0\}$ *be a fractional Brownian motion with the Hurst index* H*.*
Then, $\forall \omega \in \Omega, \forall T > 0, \forall \delta > 0, \forall 0 \leq s \leq t \leq T \quad \exists C = C(T, \omega, \delta) \in \mathbb{R}$:

$$|B_t^H - B_s^H| \leq C|t-s|^{H-\delta}.$$

Moreover, for any $p > 0$ $E(C(T, \omega, \delta))^p < \infty$.

Zero hitting: $k > 0$, $H > 1/2$

Theorem 1.10

Let $k > 0, H > 1/2$. Then the process $\{Y_t, t \geq 0\}$, defined by the equation (12) (and consequently the corresponding fractional CIR process), is strictly positive a.s.

Compare with $H = 1/2$! Only for $2k \geq \sigma^2$ we have such result, otherwise CIR process approaches zero.

Proof.

By contradiction. Assume that $a > 0$ and let for some $\omega \in \Omega$ $\tau = \inf\{t > 0 : X_t = 0\} = \inf\{t > 0 : Y_t = 0\} < \infty$. For all $\varepsilon \in (0, Y_0)$ let us also introduce the last moment of hitting the level of ε before the first zero reaching:

$$\tau_\varepsilon := \sup\{t \in (0, \tau) : Y_t = \varepsilon\}.$$

Consider δ such that both inequalities $H - \delta > 1/2$ and $1 + \delta - H < 1/2$ hold. According to the definition of τ, τ_ε and Y, the following equality is true:

$$-\varepsilon = Y_\tau - Y_{\tau_\varepsilon} = \frac{1}{2}\int_{\tau_\varepsilon}^{\tau} \left(\frac{k}{Y_s} - aY_s\right) ds + \frac{\sigma}{2}\left(B_\tau^H - B_{\tau_\varepsilon}^H\right).$$

Note that on the interval (τ_ε, τ) the process $Y_s \in (0, \varepsilon)$, hence $\forall s \in (\tau_\varepsilon, \tau)$:

$$\frac{k}{Y_s} - aY_s \geq \frac{k}{\varepsilon} - a\varepsilon. \tag{20}$$

From this and Theorem 1.9, it follows that

$$\frac{\sigma}{2} C|\tau - \tau_\varepsilon|^{H-\delta} \geq \frac{\sigma}{2} \left| B^H_\tau - B^H_{\tau_\varepsilon} \right| \geq \frac{1}{2} \left(\frac{k}{\varepsilon} - a\varepsilon \right) (\tau - \tau_\varepsilon) + \varepsilon.$$

It is clear that there exists $\tilde{\varepsilon} > 0$ such that $\forall \varepsilon < \tilde{\varepsilon} : \frac{k}{\varepsilon} - a\varepsilon > \frac{k}{2\varepsilon}$. Then, by choosing an arbitrary $\varepsilon < \tilde{\varepsilon}$, we have:

$$\frac{\sigma}{2} C |\tau - \tau_\varepsilon|^{H-\delta} \geq \frac{k}{4\varepsilon} (\tau - \tau_\varepsilon) + \varepsilon. \tag{21}$$

For $x \geq 0$ consider the function

$$F_\varepsilon(x) = \frac{k}{4\varepsilon} x - \frac{\sigma}{2} C x^{H-\delta} + \varepsilon.$$

Let us show that there exists $\varepsilon^* \in (0, \tilde{\varepsilon})$ such that for all $\varepsilon < \varepsilon^*$ and for all $x > 0 : F_\varepsilon(x) > 0$. It is easy to check that $F_\varepsilon(0) = \varepsilon > 0$ and F_ε is convex on $\mathbb{R}^+ \backslash \{0\}$ (its second derivative is strictly positive on this set), so it is enough to examine the sign of the function in its critical points.

$$F'(\tilde{x}) = \frac{k}{4\varepsilon} - \frac{\sigma\,(H-\delta)}{2} C x^{H-\delta-1} = 0 \implies$$
$$\implies \tilde{x} = \left(\frac{k}{2\sigma\varepsilon C\,(H-\delta)}\right)^{1/(H-\delta-1)} =$$
$$= \left(\frac{2\sigma C\,(H-\delta)}{k}\right)^{1/(1+\delta-H)} \varepsilon^{1/(1+\delta-H)}.$$

After some calculations we get

$$F(\tilde{x}) = \frac{1}{2}\left(\frac{2(H-\delta)}{k}\right)^{\frac{H-\delta}{1+\delta-H}} (\sigma C)^{\frac{1}{1+\delta-H}} (H-\delta-1)\varepsilon^{\frac{H-\delta}{1+\delta-H}} + \varepsilon.$$

From the choice of δ it follows that $\frac{H-\delta}{1+\delta-H} > 1$, hence $\forall K \in \mathbb{R} \quad \exists \varepsilon^* > 0 :$

$$\varepsilon - K\varepsilon^{\frac{H-\delta}{1+\delta-H}} > 0, \quad \forall \varepsilon < \varepsilon^*.$$

Choosing the corresponding ε^* for

$$K := -\frac{1}{2}\left(\frac{2(H-\delta)}{k}\right)^{\frac{H-\delta}{1+\delta-H}} (\sigma C)^{\frac{1}{1+\delta-H}} (H-\delta-1)$$

and choosing an arbitrary $\varepsilon < \min\{\tilde{\varepsilon}, \varepsilon^*\}$ we obtain that

$$F_\varepsilon(x) > 0 \quad \forall x > 0.$$

However, from (21) it follows that

$$F_\varepsilon(\tau - \tau_\varepsilon) \leq 0.$$

The contradiction obtained proves the theorem for $a > 0$. If $a \leq 0$, instead of (20) the following bound can be used:

$$\frac{k}{Y_s} - aY_s \geq \frac{k}{\varepsilon}. \tag{22}$$

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