Statistical estimators for fractional processes

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Lecture 6

Parameter estimation in the fractional models

Outline

1. Parameter estimation in the mixed models via power variations
   - Description of the mixed model and mixed power variations
   - Exact calculation and asymptotic behavior of the moments of higher order of mixed power variations
   - Weak and strong limit theorems for the centered and normalized mixed power variations
   - Statistical estimation in mixed model
   - Simulations
2. Drift parameter estimation in models with mfBm
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   - Upper bounds for the incremental variances of mfBm
   - Asymptotic growth of the trajectories of mfBm with probability 1
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Now we develop parameter estimation for the models that, while being simple enough, already depart from the canons of self-similarity. They can take into account both the independence of process increments over short time intervals and the availability of memory at longer intervals. In fact, this is the simplest version of the multi-fractional Brownian motion. More precisely, we consider so called mixed fractional Brownian motion

$$M_t^H = aB_t^H + bW_t, \quad t \geq 0.$$

This process was introduced by P. Cheridito and studied from the probabilistic point of view, e.g., in [Cheridito(2001)]. As we understand, process M^H was introduced with the aim to consider models of financial markets that are simultaneously arbitrage-free and have a memory. The applications of mixed fBm to finance and network traffic as well as the existence-uniqueness problems for the SDE involving mixed fBm, were considered in many papers, see [Androshchuk and Mishura(2006), Filatova(2008), Mishura and Shevchenko(2012), Mishura(2008)]

It turned out that in these models, in a sense, the Wiener process predominates. For example, for $H > 3/4$ the mixed fBm is equivalent in measure to the Wiener process and is a semimartingale with respect to the natural filtration. However, the presence of fBm calls for the necessity of estimating the Hurst parameter and scaling parameters a and b. Taking into account numerous articles in which statistical inference for the Wiener process and fBm, considered separately, is based on the asymptotic behavior of power variations (see, e.g., [Nourdin(2012)] and [Nourdin et al.(2010)]), we obtained and present now the results concerning both the weak and the almost sure asymptotic behavior of power variations of a linear combination of independent Wiener process and fBm. Theorems on weak convergence are based on the method of conditioning. Despite incomparably more complex calculations and estimates for variances and covariances, we succeeded not only in obtaining limit theorems, but also in calculating the exact values of numerical characteristics, using the fact that the processes are independent and Gaussian. These results are of independent interest. Then they are used to construct strongly consistent parameter estimators in mixed models.

Note that the results concerning the mixed fractional models are common with G.Shevchenko. We consider the following stochastic process, that is called a mixed fractional Brownian motion,

$$M_t^H = aB_t^H + bW_t, \quad t \geq 0, \tag{1}$$

where a and b are some non-zero coefficients, W is a Wiener process, and B^H is a fBm with Hurst index $H \in (0, 1)$. Processes W and B^H are assumed to be independent.

The advantage of the model is that it combines the properties of both the memoryless process and process with memory. Moreover, due to its simple linear structure, we can investigate its properties for any $H \in (0, 1)$.

We consider statistical identification of model (1), i. e. the statistical estimation of the model parameters. The principal attention will be given to the estimation of H, though we will also present estimators for a and b. The estimators are partially based on both weak and almost sure asymptotic behavior of mixed power variations that are related to the components of the model.

Note that asymptotic behavior of power variations and, more generally, of non-linear transformations of stationary Gaussian sequences was studied in [Breuer and Major(1983), Dobrushin and Major(1979), Giraitis and Surgailis(1985), Taqqu(1975)], and statistical estimation for fBm and multifractional processes with the help of power variations, in [Benassi et al.(1998), Coeurjolly(2001), Coeurjolly(2005), Giraitis et al.(1999), Istas and Lang(1997), Kent and Wood(1997)]. Weighted power variations serving similar purposes for stochastic differential equations driven by fBm, were studied in [Nourdin(2008), Nourdin et al.(2010), Kubilius, Mishura and Ralchenko(2017)].

Concerning the parameter estimation in the mixed model, in particular, [Cai et al.(2016)] addresses the estimation of drift parameter in a model with mixed fBm. The article [Filatova(2008)] proposes an estimation procedure for a, b and H, which is based on the empirical moments of M^H; the consistency properties are not investigated, only empirical studies for $H = 0.25$ and $H = 0.75$ are presented. The article [Xiao et al.(2011)] proposes maximum likelihood estimators of parameters in the mixed model based on the observations of the process at integer points, this is so called low-frequency data. In [Achard and Coeurjolly(2010)], the authors construct several estimators based on discrete variation. They also work in the low-frequency setting, which is essentially different from the high-frequency setting we consider. In both settings, the first order difference of the observed series is a stationary sequence. However, in the low-frequency setting the covariance does not depend on the number of observations, while in the high-frequency one, the covariance structure is very different.

Mention that for $H > 1/2$, in a small scale a mixed fBm behaves like the Wiener process. Thus, the increments of Wiener process become more and more dominating as the partition becomes finer, which makes estimation of the Hurst parameter much harder in the case where $H > 1/2$.

Our main aim is the estimation of the parameters of the process (1) based on its single observation on a uniform partition of a fixed interval. As it was mentioned above, we use power variations of this process. We remark that, in contrast to the pure fractional case, there is no self-similarity property in the mixed model (1), so we cannot directly apply the results of [Breuer and Major(1983), Dobrushin and Major(1979), Giraitis and Surgailis(1985), Taqqu(1975)] on the asymptotic behavior of sums of transformed stationary Gaussian sequences.

For this reason we need to study the asymptotic behavior as $n \to \infty$ of "mixed" power variations of the form

$$\sum_{i=0}^{n-1} \left(W_{(i+1)/n} - W_{i/n}\right)^p \left(B^H_{(i+1)/n} - B^H_{i/n}\right)^r ,$$

involving increments of independent fBm B^H and Wiener process W, where $p \geq 0$, $r \geq 0$ are fixed integer parameters. For statistical purposes, in order to construct strongly consistent estimators, we mainly need the almost sure behavior of the power variations. However, we also study their weak behavior, which is of independent interest. In particular, the calculation of the numerical characteristics of the limit Gaussian distribution and distribution involving standard Rosenblatt random variable, is provided.

Let $W = \{W_t, t \geq 0\}$ be a standard Wiener process and $B^H = \{B_t^H, t \geq 0\}$ be an independent of W fBm with Hurst parameter $H \in (0,1)$, that is, a Gaussian centered process with covariance function

$$EB_s^H B_t^H = 1/2 \left(s^{2H} + t^{2H} - |t-s|^{2H} \right),$$

both of them are defined on a complete probability space $(\Omega, \mathcal{F}, \mathrm{P})$.

For a function $X\colon [0,1] \to \mathbb{R}$ and integers $n \geq 1$, we denote $\Delta_i^n X = X_{(i+1)/n} - X_{i/n}$, $i = 0, 1, \ldots, n-1$. In this section we will study the asymptotic behavior as $n \to \infty$ of the following mixed power variations

$$\sum_{i=0}^{n-1} (\Delta_i^n W)^p \left(\Delta_i^n B^H \right)^r,$$

where $p \geq 0$, $r \geq 0$ are fixed integer numbers.

Since $\Delta_i^n W$ and $\Delta_i^n B^H$ are centered Gaussian random variables with variances n^{-1} and n^{-2H} respectively, we get that

$$\mathrm{E}\left[(\Delta_i^n W)^p \left(\Delta_i^n B^H\right)^r \right] = n^{-rH-p/2} \mu_p \mu_r,$$

where for an integer $m \geq 1$

$$\mu_m = \mathrm{E}\left[\mathcal{N}(0,1)^m \right] = (m-1)!!\, \mathbf{1}_{m \text{ is even}}$$

is the mth moment of the standard Gaussian law;
$(m-1)!! = (m-1)(m-3)\dots$ is the double factorial.

In view of this, we will study centered sums of the form

$$S_n^{H,p,r} = \sum_{i=0}^{n-1} \left(n^{rH+p/2} (\Delta_i^n W)^p \left(\Delta_i^n B^H\right)^r - \mu_p \mu_r \right).$$

Denote

$$\rho_H(m) = \mathrm{E}\left[B_1^H \left(B_{m+1}^H - B_m^H \right) \right] = \frac{1}{2} \left(|m+1|^{2H} + |m-1|^{2H} - 2\,|m|^{2H} \right) \tag{2}$$

the covariance of the so-called fractional Gaussian noise $\left\{ B_{k+1}^H - B_k^H \right\}$.

The following lemma contains an auxiliary result for calculating the characteristics of limit distributions.

Lemma 1.1

If ξ, η are centered and jointly Gaussian with unit variance and covariance ρ, then for any $r \geq 1$,

$$\mathsf{E}\left[\xi^r \eta^r\right] = \sum_{l=0}^{r/2} \frac{(r!)^2}{(2l)!((r-2l)!!)^2} \rho^{2l},$$

if r is even, and

$$\mathsf{E}\left[\xi^r \eta^r\right] = \sum_{l=0}^{(r-1)/2} \frac{(r!)^2}{(2l+1)!((r-2l-1)!!)^2} \rho^{2l+1},$$

if r is odd.

Now we can control the limit behavior of the variance of $S_n^{H,p,r}$ for different values of H.

Proposition 1

1. *If p and r are even, $r \geq 2$, then*
   - (i) *for $H \in (0, 3/4)$*

$$\mathsf{E}\left[\left(S_n^{H,p,r}\right)^2\right] \sim \left(\sigma_{H,r}^2 \mu_p^2 + \sigma_{p,r}^2\right) n, \ \ n \to \infty,$$

where $\sigma_{H,r}^2 = \sum_{l=1}^{r/2} \frac{(r!)^2}{(2l)!((r-2l)!!)^2} \sum_{m=-\infty}^{\infty} \rho_H(m)^{2l}$,
$\sigma_{p,r}^2 = \mu_{2r}\left(\mu_{2p} - \mu_p^2\right)$;

   - (ii) *for $H = 3/4$*

$$\mathsf{E}\left[\left(S_n^{3/4,p,r}\right)^2\right] \sim \sigma_{3/4,r}^2 \mu_p^2 \, n \log n, \ \ n \to \infty,$$

where $\sigma_{3/4,r} = 3r(r-1)!!/8$;

   - (iii) *for $H \in (3/4, 1)$*

$$\mathsf{E}\left[\left(S_n^{H,p,r}\right)^2\right] \sim \sigma_{H,r}^2 \mu_p^2 \, n^{4H-2}, \quad n \to \infty,$$

where $\sigma_{H,r}^2 = \frac{H^2(2H-1)r^2((r-1)!!)^2}{2(4H-3)}$.

Proposition 2

2. *If p is odd and $r \geq 1$ is arbitrary, then for any $H \in (0,1)$*

$$\mathrm{E}\left[\left(S_n^{H,p,r}\right)^2\right] = n\mu_{2p}\mu_{2r}.$$

3. *If p is even and r is odd, then*
   1. *for $H \in (0,1/2]$*

$$\mathrm{E}\left[\left(S_n^{H,p,r}\right)^2\right] \sim n\left(\sigma_{H,r}^2\mu_p^2 + \sigma_{p,r}^2\right), \quad n \to \infty,$$

where $\sigma_{H,1} = \mathbf{1}_{H=1/2}$,

$$\sigma_{H,r}^2 = \sum_{l=1}^{(r-1)/2} \frac{(r!)^2}{(2l+1)!((r-2l-1)!!)^2} \sum_{m=-\infty}^{\infty} \rho_H(m)^{2l+1} + (r!!)^2 \mathbf{1}_{H=\frac{1}{2}}, \; r \geq 3;$$

   2. *for $H \in (1/2,1)$,*

$$\mathrm{E}\left[\left(S_n^{H,p,r}\right)^2\right] \sim n^{2H}\mu_p^2\mu_{r+1}^2, \quad n \to \infty.$$

The following theorem summarizes the weak limit behavior of $S_n^{H,p,r}$.

Theorem 1.2

We have the following weak convergence, with the variances defined in Proposition 1.

1. *If p and r are even, $r \geq 2$, then*
   - (i) *for $H \in (0, 3/4)$*

$$n^{-1/2} S_n^{H,p,r} \xrightarrow{d} \mathcal{N}(0, \sigma_{H,r}^2 \mu_p^2 + \sigma_{p,r}^2), \ \ n \to \infty; \tag{3}$$

   - (ii) *for $H = 3/4$*

$$\frac{S_n^{3/4,p,r}}{\sqrt{n \log n}} \xrightarrow{d} \mathcal{N}(0, \sigma_{3/4,r}^2 \mu_p^2), \ \ n \to \infty; \tag{4}$$

   - (iii) *for $H \in (3/4, 1)$*

$$n^{1-2H} S_n^{H,p,r} \xrightarrow{d} \sigma_{H,r} \mu_p \zeta_{2H-1}, \quad n \to \infty, \tag{5}$$

     where ζ_{2H-1} is the standard Rosenblatt random variable with Hurst parameter $2H - 1$.

Theorem 1.3

2. *If p is odd and $r \geq 1$ is arbitrary, then for any $H \in (0,1)$*

$$n^{-1/2} S_n^{H,p,r} \xrightarrow{d} \mathcal{N}(0, \mu_{2p}\mu_{2r}). \tag{6}$$

3. *If p is even and r is odd, then*
   1. *for $H \in (0, 1/2]$*

$$n^{-1/2} S_n^{H,p,r} \xrightarrow{d} \mathcal{N}\left(0, \sigma^2_{H,r}\mu_p^2 + \sigma^2_{p,r}\right), \quad n \to \infty; \tag{7}$$

   2. *for $H \in (1/2, 1)$*

$$n^{-H} S_n^{H,p,r} \xrightarrow{d} \mathcal{N}\left(0, \mu_p^2 \mu_{r+1}^2\right), \quad n \to \infty. \tag{8}$$

Remark 1

For $r = 0$ we have the pure Wiener case, so for any $H \in (0, 1)$

$$n^{-1/2} S_n^{H,p,r} \xrightarrow{d} \mathcal{N}\left(0, \mu_{2p} - \mu_p^2\right), \quad n \to \infty.$$

Also note that in the case $p = 0$, $r = 1$ the limit variance in (7) vanishes. Obviously, in this case

$$n^{-H} S_n^{H,0,1} = B_1^H,$$

so it has the standard normal distribution.

The next result explains the almost sure behavior of $S_n^{H,p,r}$.

Theorem 1.4

Let $\varepsilon > 0$ be arbitrary. Then a. s.:

1. *If $r = 0$, then $S_n^{H,p,r} = o(n^{1/2+\varepsilon})$, $n \to \infty$.*
2. *If p and $r \geq 2$ are even, then*
   - (i) *for $H \in (0, 3/4]$ $S_n^{H,p,r} = o(n^{1/2+\varepsilon})$, $n \to \infty$.*
   - (ii) *for $H \in (3/4, 1)$ $S_n^{H,p,r} = o(n^{2H-1+\varepsilon})$, $n \to \infty$.*
3. *If p is odd and $r \geq 1$ is arbitrary, then for any $H \in (0, 1)$ $S_n^{H,p,r} = o(n^{1/2+\varepsilon})$, $n \to \infty$.*
4. *If p is even and r is odd, then*
   - (i) *for $H \in (0, 1/2]$ $S_n^{H,p,r} = o(n^{1/2+\varepsilon})$, $n \to \infty$.*
   - (ii) *for $H \in (1/2, 1)$ $S_n^{H,p,r} = o(n^{H+\varepsilon})$, $n \to \infty$.*

In particular, for any $H \in (0, 1)$ the following version of the ergodic theorem takes place: $n^{-1} S_n^{H,p,r} \to 0$ a. s., $n \to \infty$.

Now we turn to the question of parametric estimation in the mixed model

$$M_t^H = aB_t^H + bW_t, \quad t \in [0, T], \tag{9}$$

where a, b are non-zero numbers, which we assume to be positive, without loss of generality. Our primary goal is to construct a strongly consistent estimator for the Hurst parameter H, given a single observation of M^H.

Basing on the method of relative entropies, it was established in [Cheridito(2001)] that for $H \in (3/4, 1)$ the measure induced by M^H in $C[0, T]$ is equivalent to that of bW. Therefore, the property of almost sure convergence in this case is independent of H. Consequently, no strongly consistent estimator for $H \in (3/4, 1)$ based on a single observation of M^H exists. Denote $\Delta_i^n X = X_{T(i+1)/n} - X_{Ti/n}$ and

$$V_n^{H,p,r} = \sum_{i=0}^{n-1} (\Delta_i^n W)^p \left(\Delta_i^n B^H\right)^r.$$

Consider the quadratic variation of M^H, i. e.

$$V_n^{H,2} := \sum_{i=0}^{n-1} \left(\Delta_i^n M^H\right)^2 = a^2 V_n^{H,0,2} + 2ab V_n^{H,1,1} + b^2 V_n^{H,2,0}.$$

Note that $V_n^{H,2}$ depends only on the observed process but not on H. We use this notation to specify the distribution. Namely, we will use it to refer to the limit behavior of the quadratic variation for a specified value of the Hurst parameter H.

By Theorem on strong convergence, we have that

$$V_n^{H,0,2} \sim T^{2H} n^{1-2H}, \quad V_n^{H,2,0} \to T, \quad V_n^{H,1,1} = o(n^{1/2-H}), \quad n \to \infty.$$

Therefore, the asymptotic behavior of $V_n^{H,2}$ depends on whether $H < 1/2$ or not.

Precisely, for $H \in (0, 1/2)$,

$$V_n^{H,2} \sim a^2 T^{2H} n^{1-2H}, \quad n \to \infty, \tag{10}$$

so the quadratic variation behaves similarly to that of a scaled fBm.

For $H \in (1/2, 1)$,

$$V_n^{H,2} \to b^2 T, \quad n \to \infty, \tag{11}$$

so the quadratic variation behaves similarly to that of a scaled Wiener process.

Let us consider the cases $H < 1/2$ and $H > 1/2$ individually in more detail.

We have seen above that this case is similar to the pure fBm case. Unsurprisingly, the same estimators work, which is precisely stated below.

Theorem 1.5

For $H \in (0, 1/2)$, the following statistics

$$\widehat{H}_k = \frac{1}{2}\left(1 - \frac{1}{k}\log_2 V_{2^k}^{H,2}\right)$$

and

$$\widetilde{H}_k = \frac{1}{2}\left(\log_2 \frac{V_{2^{k-1}}^{H,2}}{V_{2^k}^{H,2}} + 1\right)$$

are strongly consistent estimators of the Hurst parameter H.

Remark 2

At first sight, there is no clear advantage of $\widehat{H}_k$ or $\widetilde{H}_k$. But a careful analysis shows that $\widetilde{H}_k$ is better. Indeed,

$$\widehat{H}_k = H - \frac{\log_2 a + H \log_2 T}{k} + o\left(k^{-1}\right), \quad k \to \infty, \tag{12}$$

while

$$\widetilde{H}_k = H + O\left(2^{k(2H-1)}\right) + o\left(2^{k(-1/2+\varepsilon)}\right), \quad k \to \infty. \tag{13}$$

Now it is absolutely clear that $\widetilde{H}_k$ performs much better (unless one hits the jackpot by having $aT^H = 1$).

Now we turn to the question of asymptotic normality of the estimators. Note that in the purely fractional case, the estimator $\widetilde{H}_k$ is asymptotically normal for all $H \in (0, 3/4)$. In the mixed case, the asymptotic normality ends at $H = 1/4$.

Proposition 3

For $H \in (0, 1/4)$,

$$2^{k/2}\left(\widetilde{H}_k - H\right) \xrightarrow{d} \mathcal{N}\left(0, (\sigma'_H)^2\right), \quad k \to \infty,$$

where $\sigma'_H = \frac{1}{\sqrt{2}\log 2}\left(\rho'_{H,0} + 2\sum_{m=1}^{\infty}\rho'_{H,m}\right)^{1/2}$,

$$\rho'_{H,m} = \mathsf{E}\Bigg[\left(\left(B_1^H\right)^2 - 2^{2H-1}\left(B_{\frac{1}{2}}^H\right)^2 - 2^{2H-1}\left(B_1^H - B_{\frac{1}{2}}^H\right)^2\right)$$
$$\times\left(\left(B_{m+1}^H - B_m^H\right)^2 - 2^{2H-1}\left(B_{m+\frac{1}{2}}^H - B_m^H\right)^2 - 2^{2H-1}\left(B_{m+1}^H - B_{m+\frac{1}{2}}^H\right)^2\right)\Bigg].$$

Now let $H \in (1/4, 1/2)$. (We omit $H = 1/4$ for two reasons: first, it is hard to distinguish this case statistically from $H \neq 1/4$; second, in this case it is shown exactly as in Proposition above that $2^{k/2}(\widetilde{H}_k - H)$ converges to a non-central limit law.) In this case neither $\widehat{H}_k$ nor $\widetilde{H}_k$ is asymptotically normal. In fact, a careful analysis of the proof of Proposition above shows that $2^{(1-2H)k}(\widetilde{H}_k - H)$ converges to some constant. Nevertheless, it is possible to construct an asymptotically normal estimator by canceling this constant out. To this end, one has to consider

$$U_k^{H,2} = V_{2^{k-1}}^{H,2} - V_{2^k}^{H,2}$$

instead of $V_{2^k}^{H,2}$. For well-definiteness we introduce the notation

$$\log_{2+} x = \begin{cases} \log_2 x, & x > 0, \\ 0, & x \leq 0. \end{cases}$$

Theorem 1.6

For $H \in (0, 1/2)$, *the statistic*

$$\widetilde{H}_k^{(2)} = \frac{1}{2}\left(\log_{2+}\frac{U_{k-1}^{H,2}}{U_k^{H,2}} + 1\right)$$

is a strongly consistent estimator of H, *moreover, for any* $\varepsilon > 0$,

$$\widetilde{H}_k^{(2)} = H + o\left(2^{k(-1/2+\varepsilon)}\right), \quad k \to \infty. \tag{14}$$

Theorem 1.7

The estimator $\widetilde{H}_k^{(2)}$ *is asymptotically normal:*

$$2^{k/2}\left(\widetilde{H}_k^{(2)} - H\right) \xrightarrow{d} \mathcal{N}\left(0, (\sigma''_H)^2\right), \quad k \to \infty,$$

with

$$\sigma''_H = \frac{1}{(1-2^{2H-1})\log 2}\left(\rho''_{H,0} + 2\sum_{m=1}^{\infty}\rho''_{H,m}\right)^{1/2},$$

$$\rho''_{H,m} = \mathsf{E}\Big[\Big(s_0^1 - (c_H+1)\big(s_0^{1/2} + c_H(s_{1/2}^{1/2}\big) + c_H\big(s_0^{1/4} + s_{1/4}^{1/4} + s_{1/2}^{1/4} + s_{3/4}^{1/4}\big)\Big)$$

$$\times\Big(s_m^1 - (c_H+1)\big(s_m^{1/2} + s_{m+1/2}^{1/2}\big) + c_H\big(s_m^{1/4} + s_{m+1/4}^{1/4} + s_{m+1/2}^{1/4} + s_{m+3/4}^{1/4}\big)\Big)\Big];$$

here $s_t^h = \left(B_{t+h}^H - B_t^H\right)^2$, $c_H = 2^{2H-1}$.

Remark 3

Despite $\widetilde{H}_k^{(2)}$ has asymptotically a better rate of approximation than $\widetilde{H}_k$ for $H \in (1/4, 1/2)$, its the asymptotic variance is high; it is practically useless for $k \leq 10$.

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Despite $\widetilde{H}_k^{(2)}$ has asymptotically a better rate of approximation than $\widetilde{H}_k$ for $H \in (1/4, 1/2)$, its the asymptotic variance is high; it is practically useless for $k \leq 10$.

Now we turn to estimation of the scale coefficients a and b. As it is known from [van Zanten(2007)], for $H \in (0, 1/4)$ the measure induced by M^H in $C[0, T]$ is equivalent to that of aB^H. This not only gives another explanation why the results for $H \in (0, 1/4)$ are essentially the same as for fBm alone, but also has another important consequence: for $H \in (0, 1/4)$ it is not possible to estimate b consistently.

Proposition 4

For $H \in (0, 1/2)$, the statistic

$$\widetilde{a}_k^2 = 2^{k(2\widetilde{H}_k - 1)} T^{-2\widetilde{H}_k} V_{2^k}^{H,2}$$

is a strongly consistent estimator of a^2.
For $H \in (1/4, 1/2)$ the statistic

$$\widetilde{b}_k^2 = \frac{2^{1-2\widetilde{H}_k^{(2)}} V_{2^{k-1}}^{H,2} - V_{2^k}^{H,2}}{\left(2^{1-2\widetilde{H}_k^{(2)}} - 1\right) T}$$

is a strongly consistent estimator of b^2.

Now we move to the case $H \in (1/2, 1)$. In view of (11), both $\widehat{H}_k$ and $\widetilde{H}_k$ converge to $1/2$ for $H \in (1/2, 1)$, so they are not suitable for estimating H. The solution is to use $U_k^{H,2} = V_{2^{k-1}}^{H,2} - V_{2^k}^{H,2}$, rather than $V_{2^k}^{H,2}$, for the construction of estimators. The resulting estimators work also for $H \in (0, 1/2)$.

Theorem 1.8

For $H \in (0, 1/2) \cup (1/2, 3/4)$, the statistics

$$\widehat{H}_k^{(2)} = \frac{1}{2}\left(1 - \frac{1}{k}\log_{2+} U_k^{H,2}\right)$$

and

$$\widetilde{H}_k^{(2)} = \frac{1}{2}\left(\log_{2+} \frac{U_{k-1}^{H,2}}{U_k^{H,2}} + 1\right)$$

are strongly consistent estimators of the Hurst parameter H.

Remark 4

We will see in Simulations that $\widetilde{H}_k^{(2)}$ performs very poorly, and $\widehat{H}_k^{(2)}$ performs somewhat better, despite having worse asymptotic rate of convergence.

As in the case $H \in (0, 1/2)$, the estimator $\widetilde{H}_k^{(2)}$ is asymptotically normal for $H \in (1/2, 3/4)$; however, the limit Gaussian law comes out of the quadratic variation of the Wiener process, so the convergence rate is different, and the expression for the asymptotic variance is much simpler.

Theorem 1.9

For $H \in (1/2, 3/4)$ and any $\varepsilon > 0$, the statistic

$$\widetilde{H}_k^{(2)} = \frac{1}{2}\left(\log_{2+} \frac{U_{k-1}^{H,2}}{U_k^{H,2}} + 1\right)$$

is a strongly consistent estimator of H, moreover, it satisfies

$$\widetilde{H}_k^{(2)} = H + o\left(2^{k(2H-3/2+\varepsilon)}\right), \quad k \to \infty. \tag{15}$$

It is asymptotically normal:

$$2^{k(3/2-2H)}\left(\widetilde{H}_k^{(2)} - H\right) \xrightarrow{d} \mathcal{N}\left(0, (\sigma_H'')^2\right), \quad k \to \infty,$$

with

$$\sigma_H''^2 = \frac{b^2 T^{1-2H}\sqrt{2^{4H-3}+1}}{a^2 2^{2H-1}(2^{2H-1}-1)\log 2}.$$

The estimation of the scale coefficient a is similar to the case $H \in (0, 1/2)$, but we have to use $U_k^{H,2}$ and $\widetilde{H}_k^{(2)}$ instead of $V_{2^k}^{H,2}$ and $\widetilde{H}_k$; the resulting estimator works also for $H \in (0, 1/2)$. Estimating b^2 is a lot easier, thanks to (11).

Proposition 5

For $H \in (0, 1/2) \cup (1/2, 3/4)$, the statistics

$$\hat{a}_k^2 = 2^{k\left(2\widetilde{H}_k^{(2)}-1\right)} T^{-2\widetilde{H}_k^{(2)}} \left(2^{2\widetilde{H}_k^{(2)}-1} - 1\right)^{-1} U_k^{H,2}$$

is a strongly consistent estimator of a^2.
For $H \in (1/2, 1)$, the statistics

$$\hat{b}_k^2 = \frac{V_{2^k}^{H,2}}{T}$$

is a strongly consistent estimator of b^2.

As we have already mentioned in the beginning of this section, it is impossible to make conclusions about the value of H in this case. In fact, we have

$$n^{1/2}(V_n^{H,2} - b^2 T) \xrightarrow{d} b^2 T \mathcal{N}(0,2), \quad n \to \infty.$$

Indeed, $n^{1/2}(V_n^{H,2,0} - T) \xrightarrow{d} \mathcal{N}(0, 2T^2)$, $n \to \infty$, by the classical CLT; $V_n^{H,0,2} \sim T^{2H} n^{1-2H} = o\left(n^{-1/2}\right)$, $n \to \infty$, and for any $\varepsilon > 0$ $V_n^{H,1,1} = o\left(n^{-H+\varepsilon}\right)$, $n \to \infty$, due to Theorem on strong convergence. This means that the behavior of $V_n^{H,2}$ is essentially the same as that of the quadratic power variation of Wiener process, in particular, it says nothing about H.

Nevertheless, we will study the behavior of quadratic variation in more detail in order to be able to distinguish between the cases $H < 3/4$, considered above, and $H > 3/4$ statistically.

Define

$$Z_k = \frac{2^{(k-1)/2}}{b^2 T} U_k^{H,2}.$$

Proposition 6

For $H \in (3/4, 1)$, the sequence $(Z_k, Z_{k+1}, \dots)$ converges in distribution as $k \to \infty$ to a sequence $(\zeta_1, \zeta_2, \dots)$ of independent standard Gaussian variables.

Remark 5

We emphasize a sharp contrast with the case $H \in (1/2, 3/4)$, where the sequence $\{Z_k, k \geq 1\}$ diverges to $+\infty$, hence, it eventually becomes positive. This clearly gives a possibility to distinguish statistically between cases $H \in (1/2, 3/4)$ and $H \in (3/4, 1)$. (See simulations.)

Remark 6

For $H = 3/4$, an analogue of recent Proposition can be proved, that is, $(Z_k, Z_{k+1}, \dots)$ converges in distribution as $k \to \infty$ to a sequence $(\zeta_1, \zeta_2, \dots)$ of independent Gaussian variables with unit variance. However, it can be checked that the limiting distribution now has a positive mean, namely, $\mathrm{E}\left[\,\zeta_1\,\right] = a^2 b^{-2} T^{\frac{1}{2}} \left(1 - 2^{-\frac{1}{2}}\right)$. As long as this value depends on how big is a compared to b, we might be unable to distinguish this case from $H > 3/4$. On the other hand, if b is small relative to a, it might be hard to distinguish this case from $H < 3/4$.

It was mentioned in the previous section that the performance of quadratic variation estimators in the case $H \in (1/2, 3/4)$ is not very satisfactory. One could try to improve it by considering quartic variation of M^H

$$V_n^{H,4} := \sum_{k=0}^{n-1} \left(\Delta_k^n M^H\right)^4 = \sum_{i=0}^{4} \binom{4}{i} a^i b^{4-i} V_n^{H,4-i,i}.$$

As for the quadratic variation, we have to cancel out the leading term, considering

$$U_k^{H,4} = V_{2^{k-1}}^{H,4} - 2V_{2^k}^{H,4}.$$

Theorem 1.10

The statistics

$$\widehat{H}_k^{(4)} = -\frac{1}{2k} \log_{2+} U_k^{H,4}$$

and

$$\widetilde{H}_k^{(4)} = \frac{1}{2} \log_{2+} \frac{U_{k-1}^{H,4}}{U_k^{H,4}}$$

are strongly consistent estimators of the Hurst parameter $H \in (1/2, 3/4)$ *in the mixed model* (9).

When the scale coefficients a and b are known, the estimation procedure significantly simplifies, and the quality of estimators is improved. It may seem unnatural at first glance that the scale coefficients are known while H is not. However, the case where b is known is quite natural, as we can have known white noise amplitude with unknown long-range perturbation of this white noise. The cases of known a or known both coefficients are less natural, but there is no reason to omit these cases considering only the case of known b.

Theorem 1.11

If a is known, then the statistic

$$\widehat{H}_k(a) = \frac{k + 2\log_2 a - \log_2 V_{2^k}^{H,2}}{2(k - \log_2 T)}$$

is a strongly consistent estimator of $H \in (0, 1/2)$, moreover, for any $\varepsilon > 0$,

$$\widehat{H}_k(a) = H + O\left(2^{k(2H-1)}\right) + o\left(2^{k(-1/2+\varepsilon)}\right), \quad k \to \infty.$$

If b is known, then the statistic

$$\widetilde{H}_k(b) = \frac{1}{2}\left(\log_{2+} \frac{V_{2^{k-1}}^{H,2} - b^2 T}{V_{2^k}^{H,2} - b^2 T} + 1\right)$$

is a strongly consistent estimator of $H \in (0, 3/4)$, moreover, for any $\varepsilon > 0$,

$$\widetilde{H}_k(b) = H + o\left(2^{k(-1/2+\varepsilon)}\right) + o\left(2^{k(2H-3/2+\varepsilon)}\right), \quad k \to \infty.$$

Remark 7

It can be shown that $\widehat{H}_k(a)$ is asymptotically normal for $H \in (0, 1/4)$, $\widetilde{H}_k(b)$, for $H \in (1/2, 3/4)$, $\widehat{H}_k(a, b)$, for $H \in (0, 3/4)$. This is not our main concern here, so we skip the asymptotic normality results.

Simulations

In each procedure we take $T = 3$, $a = b = 1$, $n = 2^{20}$ and use the circulant method to simulate values of B^H on the uniform partition $\{iT/n, i = 0, 1, \ldots, n\}$ of $[0, T]$. For each value of the Hurst parameter, we simulate 1000 trajectories of the fBm. Then for each estimator $\check{H}$ we compute the average $\mu\check{H}$ of 1000 obtained values and the mean square error $\sigma\check{H}$, i. e. the square root of the average of values $(\check{H} - H)^2$, and compare it to the theoretical standard deviation $\sigma_t\check{H}$, if the latter is available. Where possible, we make similar procedure for a and/or b.

Each simulation takes about 300 milliseconds on Intel Core i5-3210M processor, computing all estimators takes about 20 milliseconds.

In Table 1, we compare the estimators $\widehat{H}_{20}$, $\widetilde{H}_{20}$, $\widetilde{H}_{20}^{(2)}$ (observe that all these estimators are based on the values of fBm on the chosen partition). We also give values of the estimator $\widetilde{a}_{20}$; the estimator $\widetilde{b}_{20}$ is quite bad: 2–5 values of $\widetilde{b}_{20}^2$ out of 10 are negative, others are quite away from the true value, so we do not give its values.

Table: Values of the quadratic variation based estimators for $H \in (0, 1/2)$

H	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45
$\mu \widehat{H}_{20}$	.0460	.0921	.1381	.1841	.2301	.2760	.3215	.3656	.4055
$\sigma \widehat{H}_{20}$	.0040	.0079	.0119	.0159	.0199	.0240	.0285	.0344	.0445
$\mu \widetilde{H}_{20}$	.0500	.1000	.1500	.2002	.2504	.3013	.3535	.4077	.4612
$\sigma \widetilde{H}_{20}$	.0016	.0015	.0014	.0014	.0014	.0019	.0037	.0078	.0112
$\sigma_t \widetilde{H}_{20}$	.0015	.0015	.0014	.0014					
$\mu \widetilde{H}_{20}^{(2)}$	.0498	.1000	.1498	.2001	.2498	.2996	.3497	.4006	.4496
$\sigma \widetilde{H}_{20}^{(2)}$	.0047	.0051	.0053	.0058	.0069	.0082	.0103	.0155	.0345
$\sigma_t \widetilde{H}_{20}^{(2)}$	.0046	.0050	.0054	.0060	.0069	.0082	.0102	.0144	.0269
$\mu \widetilde{a}_{20}$	1.000	1.000	1.000	1.003	1.007	1.020	1.057	1.146	1.305
$\sigma \widetilde{a}_{20}$	.0200	.0196	.0180	.0175	.0181	.0263	.0591	.1468	.3050

The results show that the estimator $\widetilde{H}_{20}$ has consistently the best performance. For $H > 1/4$, a positive bias is visible, which is not surprising as it can be checked using the same transformations as in the proof of Proposition 3 that in this case

$$\widetilde{H}_k - H \sim (1 - 2^{2H-1}) a^{-2} b^2 T^{1-2H} 2^{(2H-1)k}, \quad k \to \infty.$$

The estimator $\widehat{H}_{20}$ underestimates all values of H by around 8 %. The underestimation follows from (12), since $aT^H > 1$. Finally, the relative error of $\widetilde{H}_{20}^{(2)}$ is larger than that of $\widetilde{H}_{20}$. The mean square error reflects the theoretical standard deviation quite good for all values of H except 0.4 and 0.45. The latter divergence from the theoretical values is not surprising. Indeed, a careful check of the proof of Proposition 3 reveals that the error of the normal approximation is of order 2^{-Hk}, which for values of H close to $1/2$ is comparable with the order $2^{-k/2}$ of the Gaussian term.

The estimator $\widetilde{a}_{20}$ is quite reliable, especially for smaller values of H; for $H > 1/4$ it has a positive bias (inherited from $\widetilde{H}_k$).

Table 2 compares estimators $\widehat{H}_{20}^{(2)}$ and $\widetilde{H}_{20}^{(2)}$ of Hurst parameter H. It also contains a "regression" estimator $\bar{H}^{(2)}$ obtained in the following way: we consider the linear regression of $\left\{\log_{2+} U_j^{H,2}, j = m, m+1, \dots, 19\right\}$ on $\{m, m+1, \dots, 19\}$, where $m = 11, 12, \dots, 15$, and take the best regression (in terms of the coefficient of determination). If $\bar{r}^{(2)}$ is the coefficient of the best linear regression, we set $\bar{H}^{(2)} = (1 - \bar{r}^{(2)})/2$. We also give the estimator $\hat{b}_{20}$. Due to uselessness of the estimator $\hat{a}_{20}$, we do not present its values.

It is clear that none of the estimators is reliable: average errors are in most cases comparable to the length of the range $(1/2, 3/4)$, so they are quite useless. Only the performance of $\widehat{H}_{20}^{(2)}$ in the range 0.575–0.7 is acceptable, but one should be aware of a positive bias.

It is interesting to note that the errors of both $\widetilde{H}_{20}^{(2)}$ and $\bar{H}^{(2)}$ explode for $H > 5/8$. We admit that we found no explanation for this phenomenon.

Table: Values of the quadratic variation based estimators for $H \in (1/2, 3/4)$

H	0.525	0.55	0.575	0.6	0.625	0.65	0.675	0.7	0.725
$\mu \widehat{H}_{20}^{(2)}$	.6099	.6068	.6152	.6279	.6433	.6615	.6785	.6789	.6713
$\sigma \widehat{H}_{20}^{(2)}$	.0850	.0569	.0403	.0284	.0197	.0172	.0328	.0683	.1067
$\mu \widetilde{H}_{20}^{(2)}$	.5234	.5440	.5784	.5926	.6150	.6377	.6535	.6250	.6517
$\sigma \widetilde{H}_{20}^{(2)}$	.1401	.1041	.1167	.1533	.2218	.3854	.6446	.7568	.8520
$\sigma_t \widetilde{H}_{20}^{(2)}$	.0905	.0824	.0999	.1365	.1989	.3021	.4723	.7540	1.2234
$\mu \bar{H}^{(2)}$	.5148	.5477	.5705	.5920	.6150	.6469	.7178	.7065	.7071
$\sigma \bar{H}^{(2)}$	.0830	.0543	.0549	.0717	.1014	.1607	.3272	.5247	.7174
$\mu \hat{b}_{20}$	1.236	1.131	1.071	1.038	1.020	1.011	1.006	1.003	1.002
$\sigma \hat{b}_{20}$	.2362	.1310	.0713	.0382	.0204	.0109	.0058	.0031	.0018

Table 3 contains values of $\left\{\left[10^4 U_k^{H,2}\right], k = 10, 11, 12, \ldots, 19\right\}$ for $H = 0.7$ and $H = 0.8$. The difference is clearly visible: for $H = 0.7$ the sequence is positive, while for $H = 0.8$ there is a plenty of negative values.

Table: Scaled values of $U_k^{H,2}$ for $H = 0.7$ and $H = 0.8$

$H = 0.7$	869	649	523	3	260	18	78	98	53	50
$H = 0.8$	665	-620	482	-475	8	-29	-104	-71	-78	-28

Table 4 contains estimators $\widehat{H}_{20}^{(4)}$ and $\widetilde{H}_{20}^{(4)}$ of Hurst parameter H, the values of H range from 0.525 to 0.725 with step 0.025. We also give a "regression" estimator $\bar{H}^{(4)}$. It is obtained in the following way: we consider the linear regression of $\left\{\log_{2+} U_j^{H,4}, j = m, m+1, \dots, 19\right\}$ on $\{m, m+1, \dots, 20\}$, where $m = 11, 12, \dots, 16$, and take the best regression (in terms of the coefficient of determination). If $\bar{r}^{(4)}$ is the coefficient of the best linear regression, we set $\bar{H}^{(4)} = -\bar{r}^{(4)}/2$.

Table: Values of the quartic variation based estimators for $H \in (1/2, 3/4)$

H	0.525	0.55	0.575	0.6	0.625	0.65	0.675	0.7	0.725
$\mu\widehat{H}_{20}^{(4)}$	.4840	.4876	.4997	.5157	.5321	.5509	.5460	.5001	.4450
$\sigma\widehat{H}_{20}^{(4)}$	.0414	.0626	.0754	.0846	.0934	.1021	.1793	.2899	.3838
$\mu\widetilde{H}_{20}^{(4)}$	.5313	.5527	.5799	.6122	.6103	.6230	.6085	.5064	.3915
$\sigma\widetilde{H}_{20}^{(4)}$	.1994	.1489	.1595	.2177	.3067	.5534	.8112	.8873	.9655
$\mu\bar{H}^{(4)}$	.5164	.5484	.5751	.6013	.6249	.7566	.8640	.7561	.5888
$\sigma\bar{H}^{(4)}$	.1531	.0749	.0905	.1309	.2300	.4827	.844	1.155	1.421

We see that the estimators based on the quartic variation are quite useless and definitely worse than those based on the quadratic variation. Again, the errors of $\widetilde{H}_{20}^{(4)}$ and $\mu\bar{H}^{(4)}$ explode for $H \geq 5/8$. In contrast to the quadratic variation case, now this phenomenon can be easily explained. The fact is that the nature of the error changes at $H = 5/8$: for $H < 5/8$, the error comes from the term $U_k^{H,0,4}$ (in the notation of the proof of Theorem 1.10), which behaves quite smoothly, but for $H \geq 5/8$, the main contribution comes from the fluctuations of $U_k^{H,4,0}$, which are much wilder.

Table 5 gives the estimators $\widehat{H}_{20}(a)$ and $\widehat{H}_{20}(a, b)$ for H from 0.05 to 0.45 with the step 0.05. Since the errors are very small, we multiply them by 100. We can see that the estimator $\widehat{H}_{20}(a)$ is comparable to $\widetilde{H}_{20}(a, b)$ for $H \le 1/4$; then it becomes worse, but it uses only the knowledge of a.

Table: Values of the estimators for $H \in (0, 1/2)$ and known scale coefficients

H	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45
$\mu\widehat{H}_{20}(a)$	.05	.1	.15	.2	.25	.3	.349	.397	.44
$100\sigma\widehat{H}_{20}(a)$	.007	.007	.006	.006	.009	.024	0.085	0.294	0.964
$\mu\widehat{H}_{20}(a, b)$	.05	.1	.15	.2	.25	.3	.35	.4	.45
$100\sigma\widehat{H}_{20}(a, b)$	.007	.007	.006	.006	.006	.006	.006	.006	.007

Table 6 contains the estimators $\widetilde{H}_{20}(b)$ and $\widetilde{H}^{(2)}_{20}(a,b)$ of Hurst parameter $H \in [1/2, 1)$. We multiply average errors by 10 to make them visible.

Table: Values of the estimators for $H \in [1/2, 3/4)$ and known scale coefficients

H	0.5	0.525	0.55	0.575	0.6	0.625	0.65	0.675	0.7	0.725
$\mu\widetilde{H}_{20}(b)$	.4999	.5249	.5501	.5749	.5999	.6248	.648	.6744	.6934	.7307
$10\sigma\widetilde{H}_{20}(b)$	.0196	.0288	.0432	.0713	.1251	.2136	.4055	.7316	1.48	2.402
$\mu\widehat{H}_{20}(a,b)$	.5	.525	.55	.575	.6	.625	.65	.6752	.7008	.7306
$10\sigma\widehat{H}_{20}(a,b)$	.0011	.0015	.0025	.0043	.0073	.0140	.0252	.0485	.0957	.2398

We see that $\widehat{H}_{20}(a,b)$ outperforms $\widetilde{H}_{20}(b)$ by a good margin, but the advantage of the latter is that it uses only the knowledge of b.

To facilitate the usage of the estimators, we summarize our findings about them.

For $H \in (0, 1/2)$, it is better to use the estimator $\widetilde{H}$ for the Hurst parameter. The estimator for the scale coefficient a is quite reliable, but always overestimates the coefficient for $H \in (1/4, 1/2)$. The estimator for b is virtually useless.

For $H \in (1/2, 3/4)$, there is no good estimator for the Hurst parameter. Only the regression estimator $\bar{H}^{(2)}$ is useful for values of H between 0.55 and 0.6, but still the error is comparable with the length of this integral. The coefficient b can be estimated efficiently, while the estimator for a is useless. Nevertheless, it is possible to construct efficient estimators for H using the knowledge of b or of the both scale coefficients.

Finally, for $H > 3/4$, the estimation of H is not possible (even the knowledge of the scale coefficients is not helpful). However, it is possible to distinguish statistically between the cases $H > 3/4$ and $H < 3/4$ by looking at the statistic $U_k^{H,2}$.

1. Parameter estimation in the mixed models via power variations
   - Description of the mixed model and mixed power variations
   - Exact calculation and asymptotic behavior of the moments of higher order of mixed power variations
   - Weak and strong limit theorems for the centered and normalized mixed power variations
   - Statistical estimation in mixed model
   - Simulations
2. Drift parameter estimation in models with mfBm
   - Multifractional Brownian motion
   - Upper bounds for the incremental variances of mfBm
   - Asymptotic growth of the trajectories of mfBm with probability 1
   - Asymptotic growth with probability 1 of the increments of mfBm
   - Linear multifractional model
   - Multifractional Ornstein–Uhlenbeck process

FBm is widely used in the modeling of long-range dependent processes in Internet traffic, stock markets, etc. However the stationarity of increments of fBm means that the behavior of it is the same at each point, and this substantially restricts the area of its application. In particular, it does not allow one to model situations, where the regularity at a point depends on the point. One way to overcome these limitations is to extend the standard fBm to mfBm. Following this approach, in the present section we investigate two statistical models with mfBm: the linear model and the multifractional Ornstein–Uhlenbeck process. For these models we propose estimators for an unknown drift parameter and prove their strong consistency. The proofs are based on the asymptotic bounds with probability 1 for the rate of the growth of the trajectories of mfBm and of some other functionals of mfBm, including increments and fractional derivatives.

Let $H\colon \mathbb{R}_+ \to (0,1)$ be a continuous function satisfying the following conditions:

($\mathbf{M}_1$) There exist constants $0 < h_1 < h_2 \le 1$ such that for any $t \ge 0$

$$h_1 \le H_t \le h_2.$$

($\mathbf{M}_2$) There exist constants $D > 0$ and $\kappa \in (0,1]$ such that for all $t \ge s > 0$

$$|H_t - H_s| \le D\,|t-s|^{\kappa}.$$

A *multifractional Brownian motion* (mfBm) with functional parameter H was introduced in [Benassi et al.(1997)]. It is defined by

$$Y_t = \int_{\mathbb{R}} \frac{e^{itu}-1}{|u|^{H_t+1/2}}\,\widetilde{W}(du), \quad t \ge 0, \tag{16}$$

where $\widetilde{W}(du)$ is the “Fourier transform” of the white noise $W(du)$, that is a unique complex-valued random measure such that for all $f \in L^2(\mathbb{R})$

$$\int_{\mathbb{R}} f(u) W(du) = \int_{\mathbb{R}} \widehat{f}(u) \widetilde{W}(du) \quad \text{a.s.},$$

see [Benassi et al.(1997), Stoev and Taqqu(2006)].

The covariance function of mfBm is given by

$$\mathsf{E} Y_s Y_t = D(H_s, H_t) \left(s^{H_s+H_t} + t^{H_s+H_t} - |s-t|^{H_s+H_t} \right),$$

where $D(x,y) = \frac{\pi}{\Gamma(x+y+1)\sin(\pi(x+y)/2)}$, see [Ayache et al.(2000)] or [Stoev and Taqqu(2006), p. 213]. Note that if H_t is a constant, then the process Y is an fBm (up to multiplicative constant).

In particular,

$$\left(\mathsf{E}\,|Y_t|^2 \right)^{1/2} = C(H_t) t^{H_t}, \tag{17}$$

where $C(H) = \left(\frac{\pi}{H\Gamma(2H)\sin(\pi H)} \right)^{1/2}$. Since the function $C(H)$ is bounded on $[h_1, h_2]$, we have under assumptions (M_1)–(M_2)

$$\left(\mathsf{E}\,|Y_t|^2 \right)^{1/2} \le K_1 t^{h_2}, \quad t \ge 1, \tag{18}$$

for some $K_1 > 0$.

Denote $h_3 = \min\{h_1, \kappa\}$, $h_4 = \max\{h_2, \kappa\}$, $h_5 = h_4 - h_3$, where h_1, h_2 and κ are the constants from assumptions (M_1)–(M_2).

Lemma 2.1

Under the assumption (M_1), there exists a constant $K_2 > 0$ such that for all $t \geq s \geq 0$

$$\mathsf{E}(Y(t) - Y(s))^2 \leq K_2 |t - s|^{2H_t} + K_2(H_t - H_s)^2 z^2(s), \tag{19}$$

where

$$z(s) = \begin{cases} s^{h_2} \left(\log^2 s + 1\right)^{1/2}, & s \geq 1, \\ 1, & 0 < s < 1; \end{cases}$$

Remark 8

It follows from the bound (19), from the fact that multifractional process is Gaussian, and from Kolmogorov's theorem that under conditions (M_1) and (M_2) the process Y with probability 1 has Hölder trajectories up to order $h_3 = \min\{h_1, \kappa\}$ on any finite interval.

Remark 9

There exist different definitions of mfBm in the literature. In particular, Peltier and Lévy Véhel [Peltier and Lévy Véhel(1995)] introduced mfBm based on the Mandelbrot–van Ness representation of fBm. This version of mfBm is often termed a *moving-average mfBm*. It has different covariance structure, see [Dobrić and Ojeda(2006)]. Another version of mfBm, a *Volterra-type mfBm*. It was studied in [Boufoussi et al.(2010), Ralchenko and Shevchenko(2010)]. The process introduced in (16) is sometimes called a *harmonizable mfBm* to distinguish it from other types of mfBm.

Assume that the Hurst function satisfies the conditions (M_1)–(M_2) and, additionally, $h_3 > 1/2$. In this case, according to Remark 8, process Y with probability 1 has Hölder trajectories up to order h_3 on any finite interval $[0, T]$. Assume that we have another process, say $Z = \{Z_t, t \in [0, T]\}$, also having Hölder trajectories up to some order h with $h + h_3 > 1$. In particular, it can be $h = h_3$. Then there exists an integral $\int_a^b Z_s dY_s$, which is the limit a.s. of the Riemann sums and has the standard properties (so called path-wise integral). This integral is defined as

$$\int_a^b Z dY := \int_a^b (\mathcal{D}_{a+}^{\alpha} Z)(x)(\mathcal{D}_{b-}^{1-\alpha} Y_{b-})(x) dx. \tag{20}$$

An evident estimate follows immediately from (20):

$$\left| \int_a^b Z\, dY \right| \leq \sup_{a \leq x \leq b} |(\mathcal{D}_{b-}^{1-\alpha} Y_{b-})(x)| \int_a^b |(\mathcal{D}_{a+}^{\alpha} Z)(x)| dx. \tag{21}$$

Remark 10

Bound (19) is inconvenient in the sense that it contains two different exponents of $|t-s|$ and therefore one should every time relate corresponding terms depending upon the value of $|t-s|$. To avoid this technical difficulty, we establish the next result.

Lemma 2.2

Assume that the Hurst function H satisfies the conditions (M_1) and (M_2). Let $a, b \in \mathbb{R}_+$, $b-a \geq 1$. Then

a. *for all $t, s \in [a, b]$ such that $|t-s| \leq 1$,*

$$\left(\mathsf{E}(Y(t)-Y(s))^2\right)^{1/2} \leq K_3 \left|t-s\right|^{h_3} z(b),$$

where $K_3 = K_2^{1/2}\left(1+D^2\right)^{1/2}$, D is the Hölder coefficient from (M_2).

b. *for all $t, s \in [a, b]$*

$$\left(\mathsf{E}(Y(t)-Y(s))^2\right)^{1/2} \leq K_3 \left|t-s\right|^{h_3} (b-a)^{h_5} z(b);$$

These results can give the maximal exponential bound for the weighted mfBm. In order to get it, introduce the following notations. Let b_k, $k \geq 0$, be a sequence such that $b_0 = 0$, $b_{k+1} - b_k \geq 1$, and let $a(t) > 0$ be an increasing continuous function such that $a(t) \to \infty$ as $t \to \infty$, $a_k = a(b_k)$.

Theorem 2.3

Let the Hurst function H satisfy the conditions (M_1) and (M_2). Assume that there exists $0 < \gamma \leq 1$ such that

$$\sum_{k=0}^{\infty} \frac{b_{k+1}^{h_2 + \frac{\gamma h_4}{h_3}} \left(\log^2 b_{k+1} + 1\right)^{\frac{\gamma}{2h_3}}}{a_k} < \infty. \tag{22}$$

Then for all $0 < \theta < 1$, $u > 0$

$$\mathrm{P}\left\{\sup_{t>0} \frac{|Y(t)|}{a(t)} > u\right\} \leq 2^{\frac{2}{h_3}-1} \exp\left\{-\frac{u^2(1-\theta)^2}{2A^2}\right\} A_8(\theta,\gamma), \tag{23}$$

Theorem 2.4

where

$$A = K_1 \sum_{k=0}^{\infty} \frac{b_{k+1}^{h_2}}{a_k}, \tag{24}$$

$$A_8(\theta,\gamma) = \exp\left\{ \frac{K_1^{1-\frac{\gamma}{h_3}} K_3^{\frac{\gamma}{h_3}}}{\gamma A} \sum_{k=0}^{\infty} \frac{b_{k+1}^{h_2+\frac{\gamma h_4}{h_3}} \left(\log^2 b_{k+1} + 1\right)^{\frac{\gamma}{2h_3}}}{a_k} \left(\frac{2^{\frac{2}{h_3}-1}}{\theta^{\frac{1}{h_3}}} \right)^{\gamma} \right\},$$

and additionally for all $u > A$

$$\mathrm{P}\left\{ \sup_{t>0} \frac{|Y(t)|}{a(t)} > u \right\} \le 2^{\frac{2}{h_3}-1} e^{\frac{1}{2}} \exp\left\{ -\frac{u^2}{2A^2} \right\} A_8 \left(1 - \sqrt{1 - \frac{A^2}{u^2}}, \gamma \right). \tag{25}$$

Here K_1 is the constant from (18)*.*

Remark 11

Theorem 2.3 remains true if the process $Y(t)$ is considered on the set $t > R$, where $R > 0$ is an arbitrary number. In this case the conditions (M_1) and (M_2) are replaced with the assumptions

$(\mathbf{M}_{1R})$ There exist constants $0 < h_{1R} < h_{2R} \leq 1$ such that for any $t > R$

$$h_{1R} \leq H_t \leq h_{2R}.$$

(One can put $h_{2R} = \sup_{t>R} H_t$.)

$(\mathbf{M}_{2R})$ There exist constants $D_R > 0$ and $\kappa \in (0, 1]$ such that for all $t \geq s > R$

$$|H_t - H_s| \leq D_R \, |t - s|^{\kappa}.$$

Theorem 2.5

Let the Hurst function H satisfy the conditions (M_{1R}) *and* (M_{2R}) *for some* $R>0$. *Then for any* $\delta>0$ *there exists a nonnegative random variable* $\xi=\xi(\delta)$ *such that for all* $t>R$

$$|Y(t)| \le \left(t^{h_{2R}+\delta} \vee 1\right)\xi \qquad a.\,s., \tag{26}$$

and there exist positive constants $C_1=C_1(\delta)$ *and* $C_2=C_2(\delta)$ *such that for all* $u>0$

$$\mathrm{P}(\xi>u) \le C_1 e^{-C_2 u^2}. \tag{27}$$

Let $A=\frac{K_1(Re)^{h_{2R}}}{1-e^{-\delta}}$. *Then for any* $\gamma \in \left(0, \frac{\delta h_{3R}}{h_{4R}} \wedge 1\right)$ *there exists a constant* $C_3>0$ *such that for all* $u>A$

$$\mathrm{P}(\xi>u) \le 2^{\frac{2}{h_{3R}}-1} e^{\frac{1}{2}} \exp\left\{-\frac{u^2}{2A^2}\right\} \exp\left\{C_3\left(1-\sqrt{1-\frac{A^2}{u^2}}\right)^{-\gamma/h_{3R}}\right\}. \tag{28}$$

Remark 12

The result of Theorem 2.5 cannot be improved substantially. Indeed, $Y(t)$ is a normal random variable, $\mathrm{E}Y(t) = 0$, $\sqrt{\mathrm{E}Y^2(t)} = K_1 t^{H_t}$. Therefore, $Y(t) = \xi K_1 t^{H_t}$, where ξ is a standard random variable. If there exists t_m such that $H(t_m) = \sup H_t$, then $Y(t_m) = \xi K_1 t^{H_{t_m}}$.

Theorem 2.6

Let the Hurst function H satisfy the conditions (M_1) and (M_2). Then for any $\delta > 0$ there exists t_δ such that for all $t > t_\delta$

$$|Y(t)| \leq \left(t^{h^*+\delta} \vee 1\right) \xi \qquad a.\,s.,$$

where $h^* = \limsup_{t\to\infty} H_t$, ξ is a nonnegative random variable such that for all $u > 0$

$$\mathrm{P}(\xi > u) \leq C_1 e^{-C_2 u^2}$$

for some positive constants $C_1 = C_1(\delta)$ and $C_2 = C_2(\delta)$.

Let $\Delta \in (0,1]$, $\mathbf{T}_\Delta = \left\{\mathbf{t} = (t_1, t_2) \in \mathbb{R}_+^2 : t_1 - \Delta \le t_2 \le t_1\right\}$. Consider the increment of mfBm $Z(\mathbf{t}) = Y(t_1) - Y(t_2)$, $\mathbf{t} \in \mathbf{T}_\Delta$. Let b_k, $k \ge 0$, be a sequence such that $b_0 = 0$, $b_{k+1} - b_k \ge 1$, and let $a(t) > 0$ be an increasing continuous function such that $a(t) \to \infty$ as $t \to \infty$, $a_k = a(b_k)$.

Theorem 2.7

Let the Hurst function H satisfy the conditions (M_1) and (M_2). Assume that there exists $0 < \gamma \le 1$ such that

$$\sum_{l=0}^{\infty} \frac{b_{l+1}^{\frac{2h_5\gamma}{h_3}} z(b_{l+1})}{a_l} < \infty. \tag{29}$$

Then for all $\theta \in (0,1)$, $\varepsilon \in (0, h_3)$ and $\lambda > 0$

$$\mathrm{E} \exp\left\{\lambda \sup_{\mathbf{t} \in \mathbf{T}_\Delta} \frac{|Z(\mathbf{t})|}{a(t_1)}\right\} \le \frac{1}{\Delta} \exp\left\{\frac{\lambda^2 A^2 \Delta^{2h_3}}{2(1-\theta)^2}\right\} A_9(\theta, \gamma, \varepsilon),$$

Theorem 2.8

where $A = K_3 \sum_{l=0}^{\infty} \frac{z(b_{l+1})}{a_l}$,

$$A_9(\theta,\gamma,\varepsilon) = 2^{\frac{2}{\varepsilon}+2} \exp\left\{ \frac{K_3}{A} \sum_{l=0}^{\infty} \frac{z(b_{l+1}) \log(b_{l+1}-b_l)}{a_l} \right\}$$

$$\times \exp\left\{ \frac{K_3}{\gamma A 4^{2\gamma} \left(1-\frac{\varepsilon}{h_3}\right)^{\frac{2\gamma}{\varepsilon}} \theta^{\frac{2\gamma}{h_3}}} \sum_{l=0}^{\infty} \frac{z(b_{l+1})(b_{l+1}-b_l)^{\frac{2h_5\gamma}{h_3}}}{a_l} \right\}.$$

Let d_k, $k \geq 0$, be a strictly decreasing sequence such that $d_0 = 1$, $d_k \downarrow 0$ as $k \to \infty$. Let $g\colon (0,1] \to (0,\infty)$ be a continuous function and g_k, $k \geq 0$, be a sequence such that $0 < g_k \leq \min_{d_{k+1} \leq t \leq d_k} g(t)$.

Theorem 2.9

Assume that the assumptions of Theorem 2.7 hold and

$$\sum_{k=0}^{\infty} \frac{d_k^{h_3} |\log d_k|}{g_k} < \infty.$$

Then for all $\theta \in (0,1)$, $\varepsilon \in (0,h_3)$ *and* $\lambda > 0$

$$I(\lambda) = \mathsf{E} \exp\left\{ \lambda \sup_{0 \leq t_2 < t_1 \leq t_2+1} \frac{|Z(\mathbf{t})|}{a(t_1) g(t_1 - t_2)} \right\} \leq \exp\left\{ \frac{\lambda^2 A^2 B^2}{2(1-\theta)^2} \right\} A_{10}(\theta, \gamma,$$

where

$$B = \sum_{k=0}^{\infty} \frac{d_k^{h_3}}{g_k}, \qquad A_{10}(\theta,\gamma,\varepsilon) = \exp\left\{ \frac{1}{B} \sum_{k=0}^{\infty} \frac{d_k^{h_3} |\log d_k|}{g_k} \right\} A_9(\theta,\gamma,\varepsilon).$$

Corollary 2.10

Let the assumptions of Theorem 2.9 hold. Then for all $\theta \in (0,1)$, $\varepsilon \in (0, h_3)$ and $u > 0$,

$$\mathrm{P}\left\{\sup_{0\le t_2<t_1\le t_2+1} \frac{|Z(\mathbf{t})|}{a(t_1)g(t_1-t_2)} > u\right\} \le \exp\left\{-\frac{u^2(1-\theta)^2}{2A^2B^2}\right\} A_{10}(\theta,\gamma,\varepsilon). \tag{30}$$

With the help of Corollary 2.10, we can now state the second main result of this section, which is the following upper bound for the asymptotic growth of the increments of mfBm with probability 1.

Theorem 2.11

For any $\varepsilon > 0$ and any $p > 2$ there exists a nonnegative random variable $\eta = \eta(\varepsilon, p)$ such that for all $0 \le t_2 < t_1 \le t_2 + 1$

$$|Z(\mathbf{t})| \le \left(t_1^{h_2+\varepsilon} \vee 1\right)(t_1 - t_2)^{h_3}\left(|\log(t_1 - t_2)|^p \vee 1\right)\eta \qquad a.\,s.,$$

and there exist positive constants $C_1 = C_1(\varepsilon, p)$ and $C_2 = C_2(\varepsilon, p)$ such that for all $u > 0$

$$\mathrm{P}(\eta > u) \le C_1 e^{-C_2 u^2}.$$

Consider the simplest linear model, namely

$$X_t = \theta t + Y_t, \quad t \geq 0,$$

where $\theta \in \mathbb{R}$ is an unknown parameter, Y_t is an mfBm with the Hurst function H_t satisfying the conditions (M_1)–(M_2) of Subsection 1. Assume that our aim is to estimate the parameter θ by the observations of X_t. Let us introduce the estimator

$$\hat{\theta}_T = \frac{X_T}{T} = \theta + \frac{Y_T}{T}.$$

Theorem 2.12

1. *The estimator $\hat{\theta}_T$ is strongly consistent as $T \to \infty$.*
2. *For all $T > 0$,*

$$\frac{T^{1-H_T}}{C(H_T)}\left(\hat{\theta}_T - \theta\right) \stackrel{d}{=} \mathcal{N}(0,1),$$

where $C(H) = \left(\frac{\pi}{H\Gamma(2H)\sin(\pi H)}\right)^{1/2}$. Consequently, a confidence interval of level $1-\alpha$ is given by

$$\hat{\theta}_T \pm \frac{C(H_T)}{T^{1-H_T}} z_{1-\alpha/2},$$

where z_p denotes the p-quantile of the standard normal distribution.

Let h_3 exceed $1/2$. In this subsection we consider the estimation of the unknown parameter $\theta \geq 0$ by observations of the process $X = \{X_t, t \geq 0\}$ that is a solution of the SDE of Langevin type,

$$X_t = x_0 + \theta \int_0^t X_s \, ds + Y_t, \tag{31}$$

where $x_0 \in \mathbb{R}$ is a known constant, $Y = \{Y_t, t \geq 0\}$ is an mfBm.

Note that the trajectories of the processes Y and consequently X are a. s. Hölder continuous up to order h_3. Therefore, path-wise integrals $\int_0^T X_s \, dX_s$ and $\int_0^T X_s \, dY_s$ are well defined. One can verify that the solution of (31) can be represented in the following form

$$X_t = e^{\theta t} \left(x_0 + \int_0^t e^{-\theta s} \, dY_s \right).$$

Using the integration-by-parts, this process can be written as follows

$$X_t = x_0 e^{\theta t} + \theta e^{\theta t} \int_0^t e^{-\theta s} Y_s \, ds + Y_t. \tag{32}$$

We call the process $X = \{X_t, t \geq 0\}$ a *multifractional Ornstein–Uhlenbeck process.*

Let, more precisely, our goal be to estimate the unknown drift parameter $\theta \in \mathbb{R}$ by the continuous-time observations on the interval $[0, T]$. Consider the estimator

$$\hat{\theta}_T = \frac{\int_0^T X_s \, dX_s}{\int_0^T X_s^2 \, ds}. \tag{33}$$

Since by (31), $dX_s = \theta X_s \, ds + dY_s$, we have that $\hat{\theta}_T$ admits the following stochastic representation

$$\hat{\theta}_T = \theta + \frac{\int_0^T X_s \, dY_s}{\int_0^T X_s^2 \, ds}.$$

Lemma 2.13

Let $\varepsilon > 0$, $T > 1$, $\theta > 0$. *Then*

$$\left| \int_0^T X_s \, dY_s \right| \le \zeta^2 T^{h_2+\varepsilon+1} e^{\theta T}, \qquad (34)$$

where $\zeta \ge 0$ *is a random variable with the following property: there exist positive constants* C_1 *and* C_2 *not depending on* T *such that for all* $u > 0$

$$\mathrm{P}(\zeta > u) \le C_1 e^{-C_2 u^2}.$$

Theorem 2.14

Let $\theta > 0$. *Then the estimator* $\hat{\theta}_T$ *is strongly consistent as* $T \to \infty$.

Theorem 2.15

Let $\theta = 0$. *Then the estimator* $\hat{\theta}_T$ *is consistent as* $T \to \infty$.

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