Exercise 1 (Scalas3.pdf)

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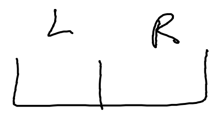

APERIODIC EHRENFEST URN

$n = 4$ Total number of balls

SYSTEM STATE

$\underline{Y}^{(4)} = (K, 4-K)$

K = # balls on L urn

$K_0, K = 0, 1, 2, 3, 4$

$\underline{Y}_0^{(4)} = (K_0, 4-K_0)$

K_0 = # balls on L urn in the starting state

TRANSITION PATH:

D_1, D_2 conditioned on $\underline{Y}_0^{(4)}$

C_1, C_2 independent

4 possible constr. $C_1 C_2$

1st	2nd	
C_L	C_L	$1/4$
C_R	C_R	$1/4$
C_R	C_L	$1/4$
C_L	C_R	$1/4$

$|S| = 5$

Aim: Transition matrix W_2 $(m=2)$ 5×5

$b_{0,j} = (b_{0,0}, b_{0,1}, b_{0,2}, b_{0,3}, b_{0,4})$

$K_0 = 0$ K

$\underline{Y}_0^{(4)} = (0, 4)$ (L R) $\xrightarrow[1]{D_R, D_R}$ $\underline{Y}^{(2)} = (0, 2)$ (L R)

C_L, C_L ($1/4$) $\to$ $\underline{Y}^{(4)} = (2, 2)$

C_R, C_R ($1/4$) $\to$ $\underline{Y}^{(4)} = (0, 4)$

$\frac{1}{2}$ C_L, C_R / C_R, C_L $\to$ $\underline{Y}^{(4)} = (1, 3)$

$$b_{0,j} = \left(\underset{b_{0,0}}{\frac{1}{4}}, \underset{b_{0,1}}{\frac{1}{2}}, \underset{b_{0,2}}{\frac{1}{4}}, \underset{b_{0,3}}{0}, \underset{b_{0,4}}{0}\right)$$

$b_{1,j} = (b_{1,0}, b_{1,1}, b_{1,2}, b_{1,3}, b_{1,4})$

$\underline{Y}_0^{(4)} = (1, 3)$ (L)

$\xrightarrow[\frac{1}{4} \cdot \frac{3}{3} = \frac{1}{4}]{D_L, D_R}$ $\underline{Y}^{(2)} = (0, 2)$ (L R)

$\xrightarrow[\frac{3}{4} \cdot \frac{1}{3} = \frac{1}{4}]{D_R, D_L}$

C_L, C_L ($1/4$) $\to$ $\underline{Y}^{(4)} = (2, 2)$

C_R, C_R ($1/4$) $\to$ $\underline{Y}^{(4)} = (0, 4)$

$\frac{1}{2}$ C_L, C_R / C_R, C_L $\to$ $\underline{Y}^{(4)} = (1, 3)$

$D_R \; D_R \quad \frac{3}{4}\cdot\frac{2}{3} = \frac{1}{2}$

$Y^{(2)} = (1,1)$

$C_L C_L \;\; 1/4 \quad Y^{(4)} = (3,1)$

$C_R C_R \;\; 1/4 \quad Y^{(4)} = (1,3)$

$\frac{1}{2} \;\; C_L C_R, \; C_R C_L \quad Y^{(4)} = (2,2)$

$$b_{1,j} = \left(\underset{b_{1,0}}{\frac{1}{8}},\; \underset{b_{1,1}}{\frac{3}{8}},\; \underset{b_{1,2}}{\frac{3}{8}},\; \underset{b_{1,3}}{\frac{1}{8}},\; \underset{b_{1,4}}{0} \right)$$

$\frac{1}{4}\cdot\frac{1}{4}\cdot 2 \qquad \frac{1}{4}\cdot\frac{1}{4}\cdot 2 + \frac{1}{2}\cdot\frac{1}{4} \qquad \frac{1}{4}\cdot\frac{1}{4}\cdot 2 + \frac{1}{2}\cdot\frac{1}{2}$

BALANCE EQUATIONS

$$\frac{b_{i,j}}{b_{j,i}} = \frac{\pi_j}{\pi_i} \qquad \pi_k = \binom{4}{k}\frac{1}{2^4}$$

$$\pi_0 = \frac{1}{16},\; \pi_1 = \frac{1}{4},\; \pi_2 = \frac{3}{8},\; \pi_3 = \frac{1}{4},\; \pi_4 = \frac{1}{16}$$

$$b_{2,j} = (b_{2,0},\, b_{2,1},\, b_{2,2},\, b_{2,3},\, b_{2,4})$$

$$b_{2,0} = b_{0,2}\cdot\frac{\pi_0}{\pi_2} = \frac{1}{4}\cdot\frac{\frac{1}{16}}{\frac{3}{8}} = \frac{1}{24}$$

$$b_{2,1} = b_{1,2}\cdot\frac{\pi_1}{\pi_2} = \frac{3}{8}\cdot\frac{\frac{1}{4}}{\frac{3}{8}} = \frac{1}{4}$$

$$b_{2,2} = 1 - b_{2,0} - b_{2,1} - b_{2,3} - b_{2,4} = \frac{5}{12} = \frac{10}{24}$$

SYMMETRY
$b_{i,j} = b_{4-i,4-j}$

$$b_{2,3} = b_{3,2}\cdot\frac{\pi_3}{\pi_2} = b_{1,2}\cdot\frac{\pi_3}{\pi_2} = \frac{3}{8}\cdot\frac{\frac{1}{4}}{\frac{3}{8}} = \frac{1}{4}$$

$$b_{2,4} = b_{4,2}\cdot\frac{\pi_4}{\pi_2} = b_{\underset{4-4,\,4-2}{0,2}}\cdot\frac{\pi_4}{\pi_2} = \frac{1}{4}\cdot\frac{\frac{1}{16}}{\frac{3}{8}} = \frac{1}{24}$$

$$W_2 = \begin{pmatrix} 1/4 & 1/2 & 1/4 & 0 & 0 \\ 1/8 & 3/8 & 3/8 & 1/8 & 0 \\ 1/24 & 1/4 & 5/12 & 1/4 & 1/24 \\ 0 & 1/8 & 3/8 & 3/8 & 1/8 \\ 0 & 0 & 1/4 & 1/2 & 1/4 \end{pmatrix}$$

symmetry

$m = 2$

$(1/2 \quad 1/2 \quad 0 \quad 0 \quad 0)$

$$W_1 = \begin{pmatrix} 1/8 & 1/2 & 3/8 & 0 & 0 \\ 0 & 1/4 & 1/2 & 1/4 & 0 \\ 0 & 0 & 3/8 & 1/2 & 1/8 \\ 0 & 0 & 0 & 1/2 & 1/2 \end{pmatrix}$$

$W_1^{(2)}$ 2 steps

$m = 1$

FOR EX

$K_0 = 0$ To reach $K = 2$

$m = 1$ $P(K_1 = 1, K_2 = 2 \mid K_0 = 0) = W_{12}(0,2) = \frac{3}{16}$

$= W_1(0,1) \cdot W(1,2)$

$m = 2$ $W_2(0,2) = \frac{1}{4}$

EVOLUTION OF EXPECTED VALUE

$K_0 = 0$

$$E(Y_t) = \sum_{K=0}^{4} K \cdot W^{(t)}(K_0, K)$$

Y_t = # balls on the 2 urn after t steps

$\text{Bin}(4, \frac{1}{2})$ $E \to n \cdot p = 4 \cdot \frac{1}{2} = 2$

NUMBER OF steps t To reach the equilibrium

$W_2^{(t)} \sim W_{inf}$

3 digit precision

$t = 13$

```
library(matrixcalc) # this library allows us to compute the power of matrices

##########################################
# Definitions of the transition matrices #
##########################################
# W1 is the matrix for unary moves, W2 for binary moves
W1<-
matrix(data=c(1/2,1/2,0,0,0,1/8,1/2,3/8,0,0,0,1/4,1/2,1/4,0,0,0,3/8,1/2,1/8,0,0,0,1/2,1/2)
nrow=5, ncol=5, byrow=TRUE)
W2<-matrix(data=c(1/4,1/2,1/4, 0, 0, 1/8, 3/8, 3/8, 1/8, 0, 1/24, 1/4, 10/24,
6/24, 1/24, 0, 1/8, 3/8, 3/8, 1/8, 0, 0, 1/4, 1/2, 1/4), nrow=5, ncol=5,
byrow=TRUE)

W12<-matrix.power(W1,2) #2-nd power of the matrix W1
W12

###############à
#Expected value

k0<-0

W1i<-matrix(0,nrow=5, ncol=5) #initialization of the i-th power of the matrix
W1
W2i<-matrix(0,nrow=5, ncol=5) #initialization of the i-th power of the matrix
W2
Mean1<-rep(0,times=30) # initialization of the mean E(Yt) in the case of unary
moves
Mean2<-rep(0,times=30) # initialization of the mean E(Yt) in the case of
binary moves
# for-cycles for the mean of Yt, that is the occupation number (number of
balls on the Left urn) after t steps
# unary moves
for (i in 1:30){
  W1i<-matrix.power(W1,i)
  Mean1[i]<-0*W1i[k0+1,1]+1*W1i[k0+1,2]+2*W1i[k0+1,3]+3*W1i[k0+1,4]+4*W1i[k0+1,5]
}
# binary moves
for (i in 1:30){
  W2i<-matrix.power(W2,i)
  Mean2[i]<-0*W2i[k0+1,1]+1*W2i[k0+1,2]+2*W2i[k0+1,3]+3*W2i[k0+1,4]+4*W2i[k0+1,5]
}
# now we plot the results
k<-seq(0,30,1)
plot(k,c(k0,Mean1), type="p", xlab="t", ylab="E(Yt)", main="Time-evolution of
the expected value")
points(k,c(k0,Mean2),type="p", pch=3)
legend(cex=1,"bottomright",c("Unary moves", "Binary moves"), pch=c(1,3))
# in both cases, the expected value tends to 2


#Approach to the equilibrium (Binomial formula (19) of scalas3)
Winf<-
matrix(data=c(1/16,1/16,1/16,1/16,1/16,1/4,1/4,1/4,1/4,1/4,3/8,3/8,3/8,3/8,3/8,1/4,1/4,1/4
nrow=5, ncol=5, byrow=FALSE)
#sufficient time t such that the power t of W2 and Winf are equal up to 3
digit precision
```

```
#by considering the maximum value of the entries given by the difference of
the matrix W2^t ans Winf
#we stop when this maximum value is 0.0000 ***with a non-zero value at the
fifth position after the dot.
for (i in 1:15){
  W2i<-matrix.power(W2,i)
  print(max(abs(W2i-Winf)))
}
```