Fractional calculus. Fractional differential equations

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2021

Lecture 5

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Outline

Introduction

In the last decade, fractional calculus has been recognized as one of the best tools to describe both long- and short-memory processes. Such models are interesting for engineers and physicists but also for pure mathematicians. The most important among such models are those described by differential equations containing fractional-order derivatives. Their evolutions behave in a much more complex way than in the classical integer-order case and the study of the corresponding theory is a hugely demanding task.

Although some results of qualitative analysis for fractional differential equations can be similarly obtained, many classical methods are hardly applicable directly to fractional differential equations. New theories and methods are thus required to be specifically developed, whose investigation becomes more challenging. Comparing with classical theory of differential equations, the researches on the theory of fractional differential equations are only on their initial stage of development.

The Riemann-Liouville fractional derivative and the Caputo fractional derivative are connected with each other by the following relations

$$ {}_a^C D_t^\alpha f(t) = {}_a D_t^\alpha f(t) - \frac{f(a)}{\Gamma(1-\alpha)}(t-a)^{-\alpha} $$

and

$$ {}_t^C D_b^\alpha f(t) = {}_t D_b^\alpha f(t) - \frac{f(b)}{\Gamma(1-\alpha)}(b-t)^{-\alpha}. $$

In particular, when $0 < \alpha < 1$ and $f \in AC([a,b], \mathbb{R}^n)$,

$$ {}_{a}^{C}D_{t}^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \left(\int_a^t (t-s)^{-\alpha} f'(s) ds \right) $$

and

$$ {}_{t}^{C}D_{b}^{\alpha} f(t) = -\frac{1}{\Gamma(1-\alpha)} \left(\int_t^b (s-t)^{-\alpha} f'(s) ds \right). $$

Neutral Equations with Bounded Delay

Introduction

Let $I_0 = [-\tau, 0], \tau > 0, t_0 \geq 0$ and $I = [t_0, t_0 + \sigma], \sigma > 0$ be two closed and bounded intervals in $\mathbb{R}$. Denote $J = [t_0 - \tau, t_0 + \sigma]$.
Let $\mathcal{C} = C(I_0, \mathbb{R}^n)$ be the space of continuous functions on I_0.
For any element $\varphi \in \mathcal{C}$, define the norm

$$\|\varphi\|_* = \sup_{\theta \in I_0} |\varphi(\theta)|.$$

If $x \in C(J, \mathbb{R}^n)$, then for any $t \in I$ define $z_t \in \mathcal{C}$ by

$$z_t(\theta) = z(t + \theta), \theta \in [-\tau, 0].$$

Consider the initial value problems (IVP for short) of fractional neutral functional differential equations with bounded delay of the form

$$\begin{cases} {}_{t_0}^{C}D_t^{\alpha}\left(x(t) - k(t, x_t)\right) = F(t, x_t), \quad a.e. \quad t \in (t_0, t_0 + \sigma], \\ x_{t_0} = \varphi, \end{cases} \tag{1}$$

where ${}_{t_0}^{C}D_t^{\alpha}$ is Caputo fractional derivative of order $0 < \alpha < 1$, $F, k : I \times \mathcal{C} \to \mathbb{R}^n$ are the given functions satisfying some assumptions that will be specified later, and $\varphi \in \mathcal{C}$.

Existence and Uniqueness

Let

$$A(\sigma,\gamma) = \{x \in C([t_0-\tau, t_0+\sigma], \mathbb{R}^n) : x_{t_0} = \varphi,\\ \sup_{t_0 \le t \le t_0+\sigma} |x(t) - \varphi(0)| \le \gamma\},$$

where σ, γ are positive constants.

Before stating and providing the main results, we introducre the following hypotheses:

(H1) $F(t,\varphi)$ is measurable with respect to t on I;

(H2) $F(t,\varphi)$ is continuous with respect to φ on $C(I_0, \mathbb{R}^n)$;

(H3) there exist $\alpha_1 \in (0,\alpha)$ and a real-valued function $m(t) \in L^{\frac{1}{\alpha_1}} I$ such that for any $x \in A(\sigma,\gamma)$, $|F(t,x)| \le m(t)$, for $t \in I_0$;

(H4) for any $x \in A(\sigma, \gamma)$, $k(t, x_t) = k_1(t, x_t) + k_2(t, x_t)$;
(H5) k_1 is continuous and for any $x', x'' \in A(\sigma, \gamma)$, $t \in I$

$$|k_1(t, x'_t) - k_1(t, x''_t)| \leq q\|x' - x''\|, where \quad q \in (0, 1);$$

(H6) k_2 is absolutely continuous and for any bounded set Λ in $A(\sigma, \gamma)$, the set $\{t \to k_2(t, x_t) : x \in \Lambda\}$ is equicontinuous in $C(I, \mathbb{R}^n)$.

Lemma 2.1

If there exist $\sigma \in (0, a)$ and $\gamma \in (0, \infty)$ such that (H1)–(H3) are satisfied, then for $t \in (t_0, t_0 + \sigma)$, IVP (1) is equivalent to the following equation

$$\begin{cases} x(t) = \varphi(0) - k(t_0, \varphi) + k(t, x_t) + \dfrac{1}{\Gamma(a)} \displaystyle\int_{t_0}^{t} (t-s)^{\alpha-1} F(s, x_s) ds, t \in I_0 \\ x_{t_0} = \varphi. \end{cases}$$

Theorem 2.2

Assume that there exist $\sigma \in (0, a)$ and $\gamma \in (0, \infty)$ such that (H1)-(H6) are satisfied. Then the IVP (1) has at least one solution on $[t_0, t_0 + \eta]$ for some positive number η.

In the case where $k_2 \equiv 0$, we get the following result.

Corollary 2.3

Assume that there exist $\sigma \in (0, a)$ and $\gamma \in (0, \infty)$ such that (H1)–(H3) hold and
(H5)' k is continuous and for any $x', x'' \in A(\sigma, \gamma)$, $t \in I$

$$|k(t, x'_t) - k(t, x''_t)| \leq q\|x' - x''\|, \textit{where} \quad q \in (0, 1);$$

Then IVP (1) has at least one solution on $[t_0, t_0 + \eta]$ for some positive number η.

In the case where $k_1 \equiv 0$, we have the following result.

Corollary 2.4

Assume that there exist $\sigma \in (0, a)$ and $\gamma \in (0, \infty)$ such that (H1)–(H3) hold and
(H6)' k is absolutely continuous and for any bounded set Λ in $A(\sigma, \gamma)$, the set $\{t \to k(t, x_t) : x \in \Lambda\}$ is equicontinuous on $C(I, \mathbb{R}^n)$.
Then IVP (1) has at least one solution on $[t_0, t_0 + \eta]$ for some positive number η.

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Thank you for your attention!