$\alpha > 0$, $\beta > -1$ $\quad a < b$

1) $\left(I^{\alpha}_{a+} f\right)(x)$, $f(t) = (t-a)^{\beta}$, $x > a$

$$\left(I^{\alpha}_{a+} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)}{(x-t)^{1-\alpha}} dt$$

$$\frac{1}{\Gamma(\alpha)} \int_a^x \frac{(t-a)^{\beta}}{(x-t)^{1-\alpha}} dt = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} (t-a)^{\beta} dt$$

$$B(x,y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt \quad \text{Beta function} \quad = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

$$[a, x] \longrightarrow [0, 1]$$

$$t - a = z \Rightarrow t = a + z$$

$$= \frac{1}{\Gamma(\alpha)} \int_0^{x-a} z^{\beta} (x-a-z)^{\alpha-1} dz$$

$$z = (x-a)u$$

$$= \frac{1}{\Gamma(\alpha)} \int_0^1 (x-a)^{\beta} u^{\beta} \left((x-a) - (x-a)u\right)^{\alpha-1} (x-a) du$$

$$= \frac{1}{\Gamma(\alpha)} (x-a)^{\beta+\alpha} \int_0^1 u^{\beta} (1-u)^{\alpha-1} du = \frac{(x-a)^{\beta+\alpha}}{\Gamma(\alpha)} B(\beta+1, \alpha)$$

$$= \frac{(x-a)^{\beta+\alpha}}{\cancel{\Gamma(\alpha)}} \frac{\Gamma(\beta+1)\cancel{\Gamma(\alpha)}}{\Gamma(\beta+1+\alpha)}$$

$$= \frac{\Gamma(\beta+1)}{\Gamma(\beta+1+\alpha)} (x-a)^{\beta+\alpha}$$

$\alpha>0,\ \beta>-1,\ a<b$

$f(t)=(b-t)^{\beta}$

$(I^{\alpha}_{b-}f)(x) \qquad x<b$

$\frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}(b-t)^{\beta}dt$

$t-x=z$

$\frac{1}{\Gamma(\alpha)}\int_0^{b-x} z^{\alpha-1}(b-x-z)^{\beta}dz$

$z=(b-x)u$

$=\frac{1}{\Gamma(\alpha)}\int_0^1 (b-x)^{\alpha-1}u^{\alpha-1}((b-x)-(b-x)u)^{\beta}(b-x)du$

$=\frac{1}{\Gamma(\alpha)}(b-x)^{\beta+\alpha}\int_0^1 u^{\alpha-1}(1-u)^{\beta}du=\frac{1}{\Gamma(\alpha)}(b-x)^{\beta+\alpha}B(\alpha,\beta+1)$

$=\frac{1}{\cancel{\Gamma(\alpha)}}(b-x)^{\beta+\alpha}\frac{\cancel{\Gamma(\alpha)}\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}$

$=\frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}(b-x)^{\beta+\alpha}$

$(I^{\alpha}_{b-}f)(x)=\frac{1}{\Gamma(\alpha)}\int_x^b\frac{f(t)}{(t-x)^{1-\alpha}}dt$

$B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt$

$[x,b]\to[0,1]$

$\alpha>0,\ \beta>0,\ a<b,\ f\in L^1(a,b),\ \forall x\in(a,b)$

$$\left(I^{\beta}_{a+}\left(I^{\alpha}_{a+}f\right)\right)(x)=\left(I^{\beta+\alpha}_{a+}f\right)(x)$$

$$\left\{\left(I^{\alpha}_{a+}f\right)(x)=\frac{1}{\Gamma(\alpha)}\int_a^x\frac{f(t)}{(x-t)^{1-\alpha}}\,dt\right\}$$

$$\left(I^{\beta}_{a+}\left(I^{\alpha}_{a+}f\right)\right)(x)=\frac{1}{\Gamma(\beta)}\int_a^x(x-t)^{\beta-1}\left(I^{\alpha}_{a+}f\right)(t)\,dt$$

$$=\frac{1}{\Gamma(\beta)}\int_a^x(x-t)^{\beta-1}\left(\frac{1}{\Gamma(\alpha)}\int_a^t\frac{f(u)}{(t-u)^{1-\alpha}}\,du\right)dt$$

$$\begin{matrix}a<u<t\\ a<t<x\end{matrix}\ \to\ \begin{matrix}u<t<x\\ a<u<x\end{matrix}$$

$$=\frac{1}{\Gamma(\beta)\Gamma(\alpha)}\int_a^x f(u)\,du\left(\int_u^x(x-t)^{\beta-1}(t-u)^{\alpha-1}\,dt\right)$$

$$g(t)=(x-t)^{\beta-1}$$

$$=\frac{1}{\Gamma(\beta)\Gamma(\alpha)}\int_a^x f(u)\,du\int_u^x(t-u)^{\alpha-1}g(t)\,dt$$

$$=\frac{1}{\Gamma(\beta)}\int_a^x f(u)\,du\,\underbrace{\frac{1}{\Gamma(\alpha)}\int_u^x(t-u)^{\alpha-1}g(t)\,dt}$$

$$I^{\alpha}_{b-}(b-t)^{\beta}=\frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}(b-x)^{\beta+\alpha}$$

$$=\frac{1}{\Gamma(\beta)}\int_a^x f(u)\,du\left(I^{\alpha}_{x-}g\right)(u).$$

$$=\frac{1}{\Gamma(\beta)}\int_a^x f(u)\,\frac{\Gamma(\beta\cancel{-1+1})}{\Gamma(\beta+\alpha\cancel{-1+1})}(x-u)^{\beta+\alpha-1}\,du$$

$$=\frac{\cancel{\Gamma(\beta)}}{\cancel{\Gamma(\beta)}}\cdot\frac{1}{\Gamma(\beta+\alpha)}\int_a^x f(u)(x-u)^{\beta+\alpha-1}\,du$$

$$=\left(I^{\beta+\alpha}_{a+}f\right)(x)$$

$$f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{\phi(t)}{(x-t)^{1-\alpha}}\, dt, \qquad x \in [a,b], \quad 0 < \alpha < 1.$$

$$f(x) = \left(I^{\alpha}_{a+} \phi\right)(x)$$

We apply $I^{1-\alpha}_{a+}$

$$\left(I^{1-\alpha}_{a+} f\right)(x) = \left(I^{1-\alpha}_{a+}\left(I^{\alpha}_{a+} \phi\right)\right)(x)$$

$$\left(I^{1-\alpha}_{a+} f\right)(x) = \left(I^{1-\cancel{\alpha}+\cancel{\alpha}}_{a+} \phi\right)(x)$$

$$\left(I^{1-\alpha}_{a+} f\right)(x) = \left(I^{1}_{a+} \phi\right)(x)$$

$$\frac{d}{dx}\left(I^{1-\alpha}_{a+} f\right)(x) = \phi(x)$$

$\alpha \geq 1$, $f \in L^1(a,b) \Rightarrow I^{\alpha}_{a^+} f \in AC([a,b])$

$[a,b]$ finite interval $\quad -\infty < a < b < +\infty$

$AC[a,b]$ space of absolutely continuous functions on $[a,b]$

$$f(x) \in AC[a,b] \Leftrightarrow f(x) = c + \int_a^x \varphi(t)\,dt , \quad \varphi \in L^1(a,b)$$

Every absolutely continuous function has a summable derivative $f'(x) = \varphi(x)$ a.e. on $[a,b]$

Moreover $c = f(a)$

$$\Rightarrow f(x) = f(a) + \int_a^x f'(t)\,dt$$

$\alpha = 1$

$I^1_{a^+} f \in AC([a,b])$ because $f \in L^1([a,b])$

$$\text{"} \int_a^x f(t)\,dt < +\infty$$

$\alpha > 1$

$$\left(I^{\alpha}_{a^+} f\right)(x) = \left(I^1_{a^+} \underbrace{\left(I^{\alpha-1}_{a^+} f\right)}_{\in L^1([a,b])}\right)(x) \in AC([a,b])$$

it is the convolution of L^1 functions

$0<\alpha<1$, $\quad f\in AC([a,b])$

- $\left(I_{a+}^{1-\alpha} f\right)(x) \in AC([a,b])$
- $\left(I_{a+}^{1-\alpha} f\right)(x) = \dfrac{1}{\Gamma(2-\alpha)}\left[f(a)(x-a)^{1-\alpha} + \displaystyle\int_a^x f'(t)(x-t)^{1-\alpha}\,dt\right]$

$$\left\{ f(x) \in AC([a,b]) \Leftrightarrow f(x) = f(a) + \int_a^x f'(t)\,dt \right\}$$

$$\left(I_{a+}^{1-\alpha} f\right)(x) = f(a)\left(I_{a+}^{1-\alpha} 1\right)(x) + \left(I_{a+}^{1-\alpha}\left(I_{a+}^{1} f'\right)\right)(x)$$

$$= f(a)\,\frac{\Gamma(1)}{\Gamma(0+1+1-\alpha)}(x-a)^{1-\alpha} + \left(I_{a+}^{2-\alpha} f'\right)(x)$$

$$= \frac{f(a)}{\Gamma(2-\alpha)}(x-a)^{1-\alpha} + \left(I_{a+}^{2-\alpha} f'\right)(x)$$

$$= \frac{f(a)}{\Gamma(2-\alpha)}(x-a)^{1-\alpha} + \frac{1}{\Gamma(2-\alpha)}\int_a^x \frac{f'(t)}{(x-t)^{1-\alpha}}\,dt$$