Periodic groups saturated by dihedral subgroups

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Involutions

An element $x \neq 1$ in a group G is called an **involution**, if $x^2 = 1$, i.e. $x = x^{-1}$

Dihedral groups.

A group is called **dihedral** if it can be generated by two distinct involutions.

Two Classical results

<u>Theorem</u> (Brauer and Fowler 1955).

There are only finitely many finite simple groups of even order in which the centralizer of an involution is isomorphic to a given group.

<u>Theorem</u> (Shunkov 1970).

If a periodic group G has an involution i with finite centralizer $C_G(i)$, then G is soluble-by-finite and so locally finite.

Lemma (The structure of a dihedral group G)

Let the dihedral group G be generated by the two distinct involutions x and y. Set $c = xy$ and $C =< c >$.

a) The subgroup C is normal and of index 2 in G,

b) If the dihedral group G is non-abelian, then C is a characteristic subgroup of G,

c) Every element of $G \setminus C$ inverts every element of C, i.e. if $g \in G \setminus C$, then $c^g = c^{-1}$,

d) Every element of $G \setminus C$ is an involution,

e) The set $G \setminus C$ is a single conjugacy class if and only if the order of C is finite and odd; it is the union of two conjugacy classes otherwise.

Locally dihedral groups

A group G is **locally (finite) dihedral** if it is the union of an infinite ascending chain of (finite) dihedral subgroups of G.

Lemma. A locally dihedral group G has a locally cylic normal subgroup C of index 2, and every element of $G \setminus C$ is an involution that inverts every element of C.

Every (locally) cyclic subgroup of G, whose order is greater than 4, bolongs to C and is normal in G, and each non-(locally) cyclic subgroup of G contains its centralizer in G.

Lemma (The structure of a locally dihedral group G)

Let $G = C \rtimes < x >$ with $x^2 = 1 \neq x$ and C locally cyclic.

a) Every non-abelian subgroup of G is locally dihedral.

b) Every abelian subgroup of G is locally cyclic or a Klein four-group.

c) If H and K are finite cyclic subgroups of G with the same order > 2, then $H = K$.

d) G contains 1, 2 or 3 conjugacy classes of involutions:

   cl K_1 consisting of all conjugates $x^g, g \in G$,
   cl K_2 consisting of a central involution z contained in C (if it exists),
   cl K_3 consisting of all conjugates $(bx)^g, g \in G$, where b is an element of C that is not a square (if it exists)

Remarks

The finite dihedral group whose cyclic normal subgroup C has order n can be defined as follows

$$D_n =< x, y; x^2 = y^2 = (xy)^n = 1 >$$

The **infinite locally dihedral 2-group** can be defined as follows

$$D_{2^\infty} =< C, i; C \simeq C(2^\infty), c^i c = 1 = i^2, c \in C >$$

Note that the **infinite quasicyclic (Prüfer)** p**-group** $C(p^\infty)$ is isomorphic with the multiplicative group of the complex p^n-roots of unity. The subgroups G_i of $C(p^\infty)$ have order p^i and form a chain

$$1 = G_0 \subset G_1 \subset G_2 \subset ...... \subset G_i \subset .....$$

Periodic groups saturated by dihedral groups

A group G is **saturated (or covered) by subgroups in a set** $\mathfrak{S}$ if every finite subgroup S of G is contained in subgroup of G which is isomorphic to a subgroup in $\mathfrak{S}$.

In the following let $\mathfrak{S}$ be the class of all finite dihedral groups.

Question. What can be said about the structure of periodic groups that are saturated by dihedral subgroups. Are they locally dihedral?

Theorem (A.K. Shlyopkin, A.G. Rubashkin, Algebra i Logika 44 (2005), 114-125).

Let G be a periodic group saturated by dihedral subgroups. In the following cases G is locally dihedral:

1. G is a Shunkov group, i.e. every pair of conjugate elements of prime order generate a finite group.
2. G is of bounded period.

If S is a Sylow 2-subgroup of G, then either S has order 2 and G is also locally dihedral, or

$G = ABC = ACB = BCA = CBA$, where A the centralizer of some involution z in the center of S, $B = O(C_G(v))$ where $v \neq z$ is an arbitrary involution in S, and $C = O(C_G(zv))$. Moreover, A is (locally) dihedral and B, C are (locally) cyclic.

<u>Lemma</u>. A locally finite group which is saturated by dihedral subgroups is locally dihedral:

If x and y are two elements of G with $o(x) > 2$ and $o(y) > 2$, then the finite group $< x, y >$ is contained in a proper finite dihedral group $D =< a > \rtimes < i >$ of G.

Then $x \in< a >$, $y \in< a >$ and so $xy = yx$.

Hence the elements of G with order more than 2 generate a locally cyclic normal subgroup H of G. The set $G \setminus H$ is non-empty and consists only of involutions.

Let $t \in G \setminus H$ a fixed and $x \in G \setminus H$ an arbitrary involution.

If $h \in H$ is of order greater than 2, then the finite subgroup $< h, x, t >$ is contained in a dihedral subgroup $D =< h_1 > \rtimes < t >$.

The definition of H implies that $h_1 \in H$. Thus all involutions of G are contained in $H \rtimes < t >$, and so $G = H \rtimes < t >$.

The counterexample

Assume there exists a periodic group G saturated by dihedral subgroups which is not locally dihedral. Then G is not locally finite. Let S be a Sylow-2-subgroup of G.

1. The centralizer $C_G(\gamma)$ of every involution γ in G is a finite or locally finite dihedral group.

2. The Sylow-2-subgroup S of G is locally dihedral.

3. All Sylow-2-subgroups of G are conjugate.

4. $N_G(S) = S$ and if two elements of S are conjugate in G, they are conjugate in S.

5. The intersection of two different Sylow subgroups of G is a cyclic group.

6. There exists a central involution τ in S. Let $A = C_G(\tau)$.

7. There exists at least one additional involution $\mu \neq \tau$ in S. Let $B = C_G(\mu)$.

8. Let S_1 be a Sylow-2-subgroup of B. Then $V =< \tau, \mu >$ of S is conjugate to S_1 in B and has order 4.

9. If S is non-abelian, then all involutions in $V- < \tau >$ are conjugate in S.

10. $G = AB$ with $A = C_G(\tau)$ and $B = C_G(\mu)$.
11. The element $\tau\mu$ is an involution with finite centralizer $C_G(\mu\tau)$.
12. G is soluble-by-finite and hence locally finite by the theorem of Shunkov.
13. You may also use theorems on factorized groups to show that G is actually soluble and locally finite
14. G is locally dihedral by the lemma above.

This contradiction proves the following

Theorem (B. Amberg, L. Kazarin, 2009).

If the infinite periodic group G is saturated by finite dihedral subgroups, then G is a locally (finite) dihedral group

Products of Chernikov groups.

Open Problem.

Is every group $G = AB$ which is the product of two Chernikov groups always a Chernikov group?

More specially,

is the group $G = AB$ a Chernikov group if the subgroups A and B are Chernikov groups such that $A/J(A)$, $B/J(B)$ **have order at most** 2**?**

Theorem (B. Amberg, L. Kazarin, Israel J. Math. 175 (2010), 363-389).

Let the group $G = AB$ be the product of two Chernikov subgroups A and B, which both have abelian subgroups A_0 and B_0 respectively with index at most 2.

Let further one of the two subgroups, A say, be of dihedral type, i.e. A contains an involution τ which inverts every element of A_0.

Then G is also a Chernikov group.

Chernikov groups.

The **finite residual** $J = J(G)$ of the group G is the intersection of all subgroups of G with finite index

$$J(G) = \bigcap G/N, N \subseteq G, |G : N| < \infty$$

A group G is a **Chernikov group** if

1. $J(G)$ is the direct product of finitely many quasicyclic (Prüfer) p-groups for finitely many primes p,
2. $G/J(G)$ is finite.

Induction parameters for Chernikov groups.

For a Chernikov group X define the parameter $\Theta(X) = (r, m)$ where

1. $r = r(X)$ is the number of quasicyclic (Prüfer) subgroups in a decomposition of the radicable abelian group $J(X)$ (the **rank** of $J(X)$)
2. $m = m(X) = |X : J(X)|$.

A linear ordering on the set of pairs (r, s) is given by $(r, s) < (r_1, s_1)$ if $r < r_1$ or $r = r_1$ and $s < s_1$.

If U is a subgroup of X, then $\Theta(U) \leq \Theta(X)$.
If $\Theta(U) = \Theta(X)$, then $U = X$.

Products of cyclic-by-finite groups

Theorem (B. A., Ya. Sysak, Arch. Math. 2008).

Let the group $G = AB$ be the product of two subgroups A and B each of which has a cyclic subgroup of index at most 2.

Then G is metacyclic-by-finite.

Remarks on the proof.

Note that up to isomorphism the infinite dihedral group is the only non-abelian group infinite group which has a cyclic subgroup of index at most 2.

The main idea of the proof is to show that the normalizer in G of an infinite cyclic subgroup of one of the factors A or B has a non-trivial intersection with the other factor.