A survey on spherical designs, with emphasis on the aspects related to finite groups

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Ischia Group Theory Conference 2010

April 16, 2010

Dedicated to the memory of Karl Gruenberg

For more details, see:
A survey on spherical designs and algebraic combinatorics on spheres, by Eiichi Bannai and Etsuko Bannai, Europ. J. Comb. 30 (2009), 1392–1425.

Spherical t-designs (Delsarte-Goethals-Seidel, 1977)

$$S^{n-1} = \{(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n \mid x_1^2 + x_2^2 + \cdots + x_n^2 = 1\}$$

$X \subset S^{n-1}$, $0 < |X| < \infty$

- X is called a spherical t-design on S^{n-1}

$$\Leftrightarrow \frac{1}{|S^{n-1}|} \int_{S^{n-1}} f(x)\, d\sigma(x) = \frac{1}{|X|} \sum_{x \in X} f(x)$$

area of S^{n-1}

for $\forall f(x) = f(x_1, \cdots, x_n)$, polynomials of degree $\leq t$

(For $r > 0$, $S^{n-1}(r) = \{(x_1, \cdots, x_n) \in \mathbb{R}^n \mid x_1^2 + x_2^2 + \cdots + x_n^2 = r^2\}$

X $(\subset S^{n-1}(r))$ is a spherical t-design on $S^{n-1}(r)$

$\Leftrightarrow \frac{1}{r} X$ $(\subset S^{n-1})$ is a spherical t-design on S^{n-1})

$\Longleftrightarrow$ Any kind of moment of X of degree $\leqslant t$ is invariant by any orthogonal transformation $\sigma \in O(n)$.

i.e., $$\sum_{x \in X} f(x) = \sum_{x \in X} f(x^\sigma).$$

for $\forall f(x) = f(x_1, \cdots, x_n)$, polynomials of degree $\leqslant t$, and for $\forall \sigma \in O(n)$.

- $X = t$-design $\Rightarrow$ $X = (t-1)$-design $\Rightarrow$ $X = (t-2)$-design. .... etc.
- $X = t$-design $\Rightarrow$ $X^\sigma = \{x^\sigma \mid x \in X\}$ is t-design, for $\forall \sigma \in O(n)$.
- $X_1 = t$-design, $X_2 = t$-design $(X_1 \cap X_2 = \emptyset)$
  $\Rightarrow$ $X_1 \cup X_2 = t$-design.

Examples of spherical t-designs

(1) $(t+1)$-vertices of a regular $(t+1)$-gon is a t-design on $S^1 (\subset \mathbb{R}^2)$

(2) vertices of regular polygons

regular polygons	# of vertices	t-design
tetrahedron	4	2
cube	8	3
octahedron	6	3
dodecahedron	20	5
icosahedron	12	5

(3) 120 vertices of regular 600-cell forms an 11-design (on $S^3 (\subset \mathbb{R}^4)$)

(4) 240 roots of type E_8 forms a 7-design (on $S^7(\sqrt{2}) \subset \mathbb{R}^8$).

(5) 196560 min. vectors of Leech lattice forms an 11-design (on $S^{23}(2) \subset \mathbb{R}^{24}$).

etc.

Questions

(1) For $n \geq 3$, when do t-designs X on S^{n-1} exist?
(For $n=2$, t-designs on S^{n-1} $(=S^1)$ exist for $\forall t \in \mathbb{N}$.)

(2) If exist, can we construct them explicitly?

(3) What is the smallest $|X|$ for t-designs X on S^{n-1}?

Remark. X = t-design on S^{n-1}

$$\Rightarrow |X| \geq \begin{cases} \binom{n-1+e}{e} + \binom{n-1+e-1}{e-1}, & \text{if } t=2e \\ 2\cdot\binom{n-1+e}{e}, & \text{if } t=2e+1 \end{cases}$$

(X is called tight t-design, if "=" holds)

Moreover, when do tight t-designs exist?

Many (but not all) spherical t-designs are obtained either as :

(i) orbits of a finite group G $(\subset O(n))$

For $G\ (\subset O(n))$ and $x \in \mathbb{R}^n$,

$X = x^G \left(= \{x^g \mid g \in G\}\right) \subset S^{n-1}(\|x\|) (\subset \mathbb{R}^n)$

or

(ii) shells of a lattice L $(\subset \mathbb{R}^n)$

$X = L_m = \{x \in L \mid x \cdot x = m\} \subset S^{n-1}(\sqrt{m}) (\subset \mathbb{R}^n)$

We want to study the previous questions for these special classes of spherical t-designs, i.e., for those which are obtained as either

orbits of a finite group, or

shells of a lattice.

Spherical t-designs which are orbits of finite group.

- $G =$ real reflection group W (on $\mathbb{R}^n$)

  e.g. $G = W(E_8)$  $|G| = 2^{14}\cdot 3^5 \cdot 5^2 \cdot 7$

  $G = W(H_4)$  $|G| = 14{,}400$

  $\exists$ exponents: $m_1, m_2, \cdots, m_n$ (with $1 = m_1 \leq m_2 \leq \cdots \cdots \leq m_n$).

  e.g. for $W(E_8)$, $(m_1, \cdots, m_8) = (1, 7, 11, 13, 17, 19, 23, 29)$.

  for $W(H_4)$, $(m_1, m_2, m_3, m_4) = (1, 11, 19, 29)$.

  For $\forall x \in S^{n-1} (\subset \mathbb{R}^n)$, $X = x^G$ becomes m_2-design.

  $\exists x_0 \in S^{n-1} (\subset \mathbb{R}^n)$ such that $X = x_0^G$ becomes m_3-design.

  So, there is an 11-design for $G = W(E_8)$ (all orbits are 7-designs)

  there is a 19-design for $G = W(H_4)$ (all orbits are 11-designs)

- For $G = \cdot 0$, all the orbits are 11-designs, while there are 15-designs which are orbits of G.

Known results

ρ_i = The i-th spherical irred. representation of $O(n)$

$$\Big(\longleftrightarrow \underbrace{\boxed{\ \ \big|\ \ \big|\ \cdots\ \big|\ \ }}_{i} \Big)$$

$$O(n) \xrightarrow{\rho_i} \mathrm{Harm}_i(\mathbb{R}^n) = \text{The space of homogeneous harmonic polynomials of degree } i \text{ on } \mathbb{R}^n$$

$$\left(\dim \mathrm{Harm}_i(\mathbb{R}^n) = \binom{n-1+i}{i} - \binom{n-1+i-2}{i-2} \right)$$

Let $G \subset O(n)$, $0 < |G| < \infty$.

- (Goethals-Seidel, 1979).

  Any orbits $X = x^G$ of G are t-designs $\Longleftrightarrow$ $(\rho_0, \rho_i)_G = 0$ for $i = 1, 2, \cdots, t$

  identity rep. of $O(n)$

  (We call such $G \subset O(n)$, t-homogeneous.)

- (Bannai, 1979)

  $\rho_i|G$ are all irreducible for $i = 1, 2, \cdots, e$ $\Longrightarrow$ G is $2e$-homogeneous

- (de la Harpe-Pache, 2004)

  $\rho_e|G$ is irreducible $\Longrightarrow$ G is $2e$-homogeneous

- (Bannai, 1984)
  $\exists f(n)$ such that $t \leq f(n)$ for $n \geq 3$, if
  $G \subset O(n)$, $X = x^G$ is a t-design for some $x \in S^{n-1} (\subset \mathbb{R}^n)$.
- (Bannai, 1984)
  $X_1 = x_1^G$ is a t_1-design, $X_2 = x_2^G$ is a t_2-design
  $\Rightarrow \quad t_1 \leq 2t_2 + 1 \quad (t_2 \leq 2t_1 + 1)$.
- Corollary
  $X = x^G$ is a t-design $\Rightarrow G = \left[\frac{t-1}{2}\right]$ homogeneous.

Conjectures

(1) $X = x^G$ is t-transitive $\Longrightarrow t \overset{\exists}{\leq} t_0$ (absolute constant, say $t_0 = 19$)

(2) $G = t$-homogeneous $\Longrightarrow t \leq 11$.

(Miezaki, 2010, proved that $G \subset O(4) \Rightarrow G$ is at most 11-homogeneous.
He uses the classification of all the finite subgroups of $O(4)$ by Conway-Smith.)

- Can one use the classification of finite simple groups to prove these conjectures ?

Spherical t-designs which are shells of lattice

$\mathrm{Min}\{x \cdot x \mid x \in L, x \neq 0\} = 2 + 2\left[\frac{n}{24}\right]$

- (Venkov, 1984) L = even unimodular $\overbrace{\text{extremal}}$ lattice in $\mathbb{R}^n$
  ($\Rightarrow n \equiv 0 \pmod 8$)

$(= \{x \in L \mid x \cdot x = 2m\})$

$\Rightarrow$ Any shell L_{2m} of L is

$$\begin{cases} 11\text{-design}, & \text{if } n \equiv 0 \pmod{24} \\ 7\text{-design}, & \text{if } n \equiv 8 \pmod{24} \\ 3\text{-design}, & \text{if } n \equiv 16 \pmod{24}. \end{cases}$$

In particular, any shell of E_8-lattice is a 7-design,
any shell of Leech lattice is an 11-design.

Is there any shell of E_8-lattice L which is an 8-design?

- (Venkov, de la Harpe, Pache, 2005)

L_{2m} = 8-design $\iff$ $\tau(m) = 0$.

Ramanujan τ-function

$$q \prod_{i=1}^{\infty} (1-q^i)^{24} = \eta(z)^{24} \quad (q = e^{2\pi i z})$$

$$= q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 - 6048q^6 - 16744q^7 + \cdots$$

$$= \sum_{m=1}^{\infty} \tau(m) \cdot q^m$$

D. H. Lehmer's conjecture $\iff$ $\tau(m) \neq 0$, $\forall m \geq 1$ $(m \in \mathbb{N})$

D.H. Lehmer's conjecture is still open!

We solved an analogous (but easier) situation.

<u>Thm</u> (Bannai-Miezaki, arXiv:0812.4643, to appear in J. Math. Soc. Japan)

- $L = \mathbb{Z}_2$-lattice $\Rightarrow$ no shell is a 4-design. (all the shells are 3-designs)
- $L = A_2$-lattice $\Rightarrow$ no shell is a 6-design (all the shells are 5-designs)

<u>Further results</u>

Bannai-Miezaki, Toy models for D.H. Lehmer's conjecture, II arXiv:1004.1520

Bannai-Miezaki-Yudin, An elementary approach to toy models for D.H. Lehmer's conjecture, arXiv:1003.4414

Remarks

- In $\mathbb{R}^2$, no 6-design which is a shell of a lattice is known.
- In $\mathbb{R}^3$, no 4-design which is a shell of a lattice is known.
- In $\mathbb{R}^n$ ($\forall n \geq 2$), no 12-design which is a shell of a lattice is known. ($\underline{\underline{t \leq 11}}$)

Compare this with the classical Assmus-Mattson theorem

no 6-design which is a shell of a code is known. $(\underline{\underline{t \leq 5}})$

Gegenral spherical t-designs (Existence and constructions)

Thm (Seymour-Zaslavsky, 1984)

For any t and any n, spherical t-designs on S^{n-1} exist.

This is an existence theorem, and explicit constructions are difficult if $n \geq 3$ and t large.

- Explicit construction is kind of known for $n=3$ by G. Kuperberg (2005).

(His method is expected to work also for $n \geq 4$, but currently it is still open.)

Thank you!