Subgroups with non-trivial Möbius number in the Alternating and Symmetric groups

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Let G be a finitely generated profinite group. We consider G as a probability space (with respect to the normalized Haar measure) and denote by $P(G,k)$ the probability that k randomly chosen elements of G generate G itself.

G is called positively finitely generated (PFG) if $P(G,k) > 0 \ \exists k \in \mathbb{N}$.

Definitions

- For each open subgroup H of G we may define

$$\mu(H,G) = \begin{cases} 1 & \text{if } H = G \\ -\sum_{H<K\leq G} \mu(K,G) & \text{otherwise.} \end{cases}$$

- For each $m \in \mathbb{N}$, let $b_m(G)$ be the number of open subgroups H with $|G:H| = m$ and $\mu(H,G) \neq 0$.

Conjectures

Conjecture (Mann, 2005)

Let G be a PFG group. Then $b_m(G)$ and $|\mu(H,G)|$ are bounded polynomially, respectively in terms of m and $|G:H|$.

This conjecture by Mann is implied by the following:

Conjecture (Lucchini, 2010)

There exists a constant c such that if X is a finite almost simple group, then $b_m(X) \leq m^c$ and $|\mu(Y,X)| \leq |X:Y|^c$ for each $m \in \mathbb{N}$ and each $Y \leq X$.

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Main theorems

We have proved that Lucchini's conjecture holds for all the Alternating and Symmetric groups:

Theorem 1

There exists an absolute constant α such that for any $n \in \mathbb{N}$, if $X \in \{\mathrm{Alt}(n), \mathrm{Sym}(n)\}$ and $m \in \mathbb{N}$, then $b_m(X) \leq m^\alpha$.

Theorem 2

There exists an absolute constant β such that for any $n \in \mathbb{N}$, if $X \in \{\mathrm{Alt}(n), \mathrm{Sym}(n)\}$ and $Y \leq X$, then $|\mu(Y, X)| \leq |X : Y|^\beta$.

Further results

These two theorems and Lucchini's reduction theorem imply:

Corollary 1

If G is a PFG *group, and for each open normal subgroup N of G all the composition factors of G/N are either abelian or Alternating groups, then*

- *there exists γ_1 such that $b_m(G) \leq m^{\gamma_1}$ for each $m \in \mathbb{N}$;*
- *there exists γ_2 such that $|\mu(H, G)| \leq |G : H|^{\gamma_2}$ for each open subgroup H of G.*

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Example

$G = \prod_n (\mathrm{Alt}(n))^n$ satisfies Mann's conjecture.

Further results

Theorem (Lucchini, 2009)

Let $G = \prod_i S_i$, where the S_i's are finite nonabelian simple groups; suppose that G is d-generated and that there exists a constant c such that: $|\mu(Y, S_i)| \leq |S_i : Y|^c$, $\forall\, i$ and $\forall\, Y \leq S_i$. Then

$$|\mu(H, G)| \leq |G : H|^{\epsilon}$$

for each open subgroup H of G, where $\epsilon = \max(d, c) + 1$.

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Corollary 2

Let $G = \prod_i A_i$, where the A_i's are Alternating groups; suppose that G is d-generated. Then

$$|\mu(H, G)| \leq |G : H|^{\max(d, \beta)+1}$$

with β as in Theorem 2.

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Example

$G = \prod_{n \geq 5} (\mathrm{Alt}(n))^{n!/8}$ is 2-generated and then, by Corollary 2, we have

$$|\mu(H, G)| \leq |G : H|^{\max(2,\beta)+1}$$

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Remark

Note that $G = \prod_{n \geq 5} (\mathrm{Alt}(n))^{n!/8}$ is not a PFG group.

Preliminaries

Let G be transitive on a finite set Γ; $\mathcal{L}_G$ is the subgroup lattice of G.

Definition

Let $H \leq G$ and let $\{\Omega_1, \ldots, \Omega_r\}$ be the orbits of H on Γ; define

$$\overline{H} := (\prod_i \mathrm{Sym}(\Omega_i)) \cap G$$

the **closure** of H in $\mathcal{L}_G$. H is said **closed** in $\mathcal{L}_G$ if and only if $H = \overline{H}$. The set $\overline{\mathcal{L}}_G := \{H \in \mathcal{L}_G \,|\, H = \overline{H}\}$ is a poset; for any $H \in \overline{\mathcal{L}}_G$, denote by $\overline{\mu}(H, G)$ the Möbius number of H in $\overline{\mathcal{L}}_G$.

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By the closure theorem of Crapo on $\mathcal{L}_G$:

$$\sum_{\substack{K \leq G \\ K \text{ transitive}}} \mu(H, K) = \begin{cases} \overline{\mu}(H, G) & \text{if } H = \overline{H} \\ 0 & \text{otherwise.} \end{cases}$$

Key step

Lemma 3

If H is a subgroup of a transitive permutation group G, then

$$\mu(H,G) = \sum_{K \in \mathcal{S}_H} \mu(K,G) g(H,K),$$

where $\mathcal{S}_H := \{K \leq G \,|\, K \text{ transitive on } \Gamma, K \geq H\}$

and $g(H,K) = \begin{cases} \overline{\mu}(H,K) & \text{if } H \text{ is closed in } \mathcal{L}_K \\ 0 & \text{otherwise.} \end{cases}$

We will apply this lemma when $G \in \{\mathrm{Sym}(n), \mathrm{Alt}(n)\}$. In particular we will consider G with two different actions:

- the natural action on the set $I_n := \{1, \ldots, n\}$,
- the action on the set $\Delta_n := \{(a,b) \,|\, 1 \leq a, b \leq n,\ a \neq b\}$ defined by $(a,b)g = (ag, bg)$. It is a transitive action.

Proof of Theorem 2

Theorem 2

There exists an absolute constant β such that for any $n \in \mathbb{N}$, if $G \in \{\mathrm{Alt}(n), \mathrm{Sym}(n)\}$ and $H \leq G$, then $|\mu(H, G)| \leq |G : H|^{\beta}$.

Let $H \leq G$. We apply Lemma 3 with respect to the natural action of G on $I_n = \{1, \ldots, n\}$:

$$\mu(H, G) = \sum_{T \in \mathcal{S}_H} \mu(T, G) g(H, T)$$

with $\mathcal{S}_H = \{T \leq G \,|\, T \text{ transitive}, T \geq H\}$. Then

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1) First of all we estimate $|g(H,T)|$. Let r be the number of orbits of H on $\{1,\ldots,n\}$; then
   - $|g(H,T)|$ is bounded in terms of r: for any $T \neq H$

$$|g(H,T)| \leq (r\,!)^2/2,$$

   - $r!$ is bounded in terms of $|G:H|$:

$$r! \leq 2 \cdot |G:H|.$$

Hence

$$|g(H,T)| \leq 2 \cdot |G:H|^2 \qquad \forall\, T \in \mathcal{S}_H.$$

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2) Then we give a bound for $|\mu(T,G)|$. Consider the action of G on $\Delta_n = \{(a,b) \mid 1 \leq a,b \leq n,\ a \neq b\}$; by applying Lemma 3, we obtain

$$|\mu(T,G)| \leq \sum_{R \in \mathcal{S}_T} |\mu(R,G)| \cdot |g(T,R)|$$

with $\mathcal{S}_T = \{R \leq G \mid R \text{ transitive on } \Delta_n, R \geq T\}$. Let t be the number of orbits of T on Δ_n; then

- As previously, $|g(T,R)| \leq (t\,!)^2/2 \leq 2 \cdot |G:T|^2$,
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Hence $\exists\,\nu$ (independent of n) such that

$$|\mu(T,G)| \leq |G:T|^{\nu} \leq |G:H|^{\nu} \qquad \forall\, T \in \mathcal{S}_H$$

Denote by s the number of $T \in \mathcal{S}_H$ such that $\mu(T,G) \neq 0$. Then

$$\sum_{T \in \mathcal{S}_H} |\mu(T,G)| \leq s \cdot |G:H|^{\nu}$$

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Aim: to bound polynomially s, in terms of $|G:H|$.

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$$t(G) \leq (n!)^d$$

We have

$$|\mu(T, G)| \leq \sum_{R \in \mathcal{S}_T} |\mu(R, G)| \cdot |g(T, R)|$$

with $\mathcal{S}_T = \{R \leq G \mid R \text{ transitive on } \Delta_n, R \geq T\}$.
Since $\mu(T, G) \neq 0 \Rightarrow \exists\, R \in \mathcal{S}_T$ such that $g(T, R) = \overline{\mu}(T, R) \neq 0$.
Then T is closed in $\mathcal{L}_R$, and

$$T = R \cap C$$

where R is 2-transitive, and C is Δ_n-closed and transitive in G.

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3) We start proving: $t(G)$ be the number of all the transitive subgroups T of G with $\mu(T,G) \neq 0$; then $\exists\, d$, independent of n, such that

$$t(G) \leq (n!)^d$$

We have

$$|\mu(T,G)| \leq \sum_{R \in \mathcal{S}_T} |\mu(R,G)| \cdot |g(T,R)|$$

with $\mathcal{S}_T = \{R \leq G \mid R \text{ transitive on } \Delta_n, R \geq T\}$.
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Let $m \in \mathbb{N}$; denote by $t_m(G)$ the number of transitive subgroups T of G such that $|G : T| = m$ and $\mu(T, G) \neq 0$. If $m \leq 2$, $t_m(G) \leq 1$. Let $m > 2$; there exists an absolute constant f such that:

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- if $m^f < n!$ (i.e. m is very "small"), then any transitive subgroup T of G, with $|G : T| = m$, is imprimitive and

$$(\mathrm{Alt}(a))^b \leq T \leq \mathrm{Sym}(a) \wr \mathrm{Sym}(b)$$

where $1 < b < a$, $ab = n$. The number of these subgroups of index m can be bounded polynomially on m.

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$$t_m(G) \leq m^\eta \qquad \forall\, m \in \mathbb{N}.$$

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Aim: to bound polynomially s, in terms of $|G : H|$.

s: number of transitive subgroups of G containing H and with non zero Möbius number.
Then

$$s \leq \sum_{m \leq |G:H|} t_m(G) \leq |G : H|^{\eta+1}.$$

Conclusion

$$|\mu(H, G)| \leq 2 \cdot |G : H|^2 \cdot |G : H|^\nu \cdot |G : H|^{\eta+1} \qquad (*)$$

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Some remarks

If we consider prime degrees, we are able to improve this result:

Theorem 4

Let p be a prime, with $p \neq 11, 23$ and $p \neq (q^d - 1)/(q - 1)$, for any (q, d), with q a prime power and $q > 4$ if $d = 2$.
If $G \in \{\mathrm{Alt}(p), \mathrm{Sym}(p)\}$ and $H \leq G$, then

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Theorem 4 leads us to formulate

Conjecture

For any $n \in \mathbb{N}$, *if* $G \in \{\mathrm{Alt}(n), \mathrm{Sym}(n)\}$ *and* $H \leq G$, *then*

$$|\mu(H, G)| \leq c \cdot |G : H| \qquad \exists\, c$$

The Reduction Theorem

Denote by $\Lambda(G)$ the set of finite monolithic groups L such that soc L is non abelian and L is an epimorphic image of G.

Theorem (Lucchini)

Let G be a PFG *group. Then the followings are equivalent.*

(1) *There exist two constants γ_1 and γ_2 such that*

$$b_m(G) \leq m^{\gamma_1} \quad \text{and} \quad |\mu(H,G)| \leq |G:H|^{\gamma_2}$$

for each $m \in \mathbb{N}$ and each open subgroup H of G.

(2) *There exist two constants c_1 and c_2 such that*

$$b_m(X_L) \leq m^{c_1} \quad \text{and} \quad |\mu(Y,X_L)| \leq |X_L:Y|^{c_2}$$

for each $L \in \Lambda(G)$, each $m \in \mathbb{N}$ and each $Y \leq X_L$.

Proof of Theorem 1

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There exists an absolute constant α such that for any $n \in \mathbb{N}$, if $G \in \{\mathrm{Alt}(n), \mathrm{Sym}(n)\}$ and $m \in \mathbb{N}$, then $b_m(G) \leq m^\alpha$.

Let $H \leq G$ with $|G : H| = m$ and $\mu(H, G) \neq 0$. We apply Lemma 3 with respect to the natural action of G on $\{1, \ldots, n\}$:

$$\mu(H, G) = \sum_{T \in \mathcal{S}_H} \mu(T, G) g(H, T)$$

with $\mathcal{S}_H = \{T \leq G \,|\, T \text{ transitive}, T \geq H\}$.
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Strategy

To bound $b_m(G)$, we have to find polynomial bounds, in terms of m:

1) for the number of closed subgroups of G with index dividing m,
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Lemma 5

Let $G \in \{\mathrm{Alt}(n), \mathrm{Sym}(n)\}$ *and denote by* $c_m(G)$ *the number of subgroups of* G *with index* m *and closed in* $\mathcal{L}_G$*. Then* $c_m(G) \leq m^4$ *for each* $m \in \mathbb{N}$*.*

Lemma 6

Let $G \in \{\mathrm{Alt}(n), \mathrm{Sym}(n)\}$ *and denote by* $t_m(G)$ *the number of transitive subgroups* T *of* G *with* $|G : T| = m$ *and* $\mu(T, G) \neq 0$*. Then there exists an absolute constant* η *such that* $t_m(G) \leq m^\eta$ *for each* $m \in \mathbb{N}$*.*

Key step

Let G be transitive on Γ. For any subgroup H of G, define

$$\mathcal{S}_H := \{K \leq G \,|\, K \text{ transitive on } \Gamma, K \geq H\} \subseteq \mathcal{L}_G.$$

Define $f, g : \mathcal{L}_G \times \mathcal{L}_G \to \mathbb{Z}$:

$$f(H, Y) = \begin{cases} \mu(H, Y) & \text{if } Y \in \mathcal{S}_H \\ 0 & \text{otherwise,} \end{cases}$$

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Then f and g satisfy the relation $g(H,X) = \sum_{\substack{Y \leq X \\ Y \in \mathcal{S}_H}} f(H,Y)$.

By the Möbius inversion formula, for any $Y \in \mathcal{S}_H$, we have

$$f(H,Y) = \sum_{\substack{X \leq Y \\ X \in \mathcal{S}_H}} \mu(X,Y) g(H,X).$$

Setting $Y = G$, we get:

Lemma 3

If H is a subgroup of a transitive permutation group G, then

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Lemma 3

If H is a subgroup of a transitive permutation group G, then

$$\mu(H,G) = \sum_{K \in \mathcal{S}_H} \mu(K,G) g(H,K).$$