The Markov-Zariski topology of an infinite group

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Dedicated to memory of Maria Silvia Lucido joint work with Daniele Toller and Dmitri Shakhmatov

1. Markov's problem 1 and 2
2. The three topologies on an infinite group
3. Problem 1 and 2 in topological terms
4. The Markov-Zariski topology of an abelian group
5. Markov's problem 3.

Markov's problem 1

Definition

A group G is *topologizable* if G admits a non-discrete Hausdorff group topology.

Problem 1. [1944]

Does there exist a (countably) infinite non-topologizable group?

- Yes (under CH): Shelah, *On a problem of Kurosh, Jonsson groups, and applications*. In *Word Problems II*. (S. I. Adian, W. W. Boone, and G. Higman, Eds.) (North-Holland, Amsterdam, 1980), pp.373–394.
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Markov's problem 2

Definition (Markov)

A subset S of a group G is called:

(a) elementary algebraic if $S = \{x \in G : a_1 x^{n_1} a_2 x^{n_2} a_3 \dots a_m x^{n_m} = 1\}$ for some natural m, integers $n_1, \dots, n_m$ and elements $a_1, a_2, \dots, a_m \in G$.

(b) algebraic, if S is an intersection of finite unions of elementary algebraic subsets.

(c) unconditionally closed, if S is closed in *every* Hausdorff group topology of G.

Every centralizer $c_G(a) = \{x \in G : axa^{-1}x^{-1} = 1\}$ is an elementary algebraic set, so $Z(G)$ is an algebraic set.

(a) $\to$ (b) $\to$ (c)

Problem 2. [1944]

Is (c) $\to$ (a) always true ?

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The Zariski topology

$\mathfrak{E}_G$ the family of elementary algebraic sets of G. $\mathfrak{A}_G$ the family of all algebraic sets of G. The *Zariski* topology $\mathfrak{Z}_G$ of G has $\mathfrak{A}_G$ as family of all closed sets.

Bryant, Roger M. *The verbal topology of a group.* J. Algebra 48 (1977), no. 2, 340–346. Wehrfritz's MR-review: This paper is beautiful, short, elementary and startling. It should be read by every infinite group theorist. The author defines on any group (by analogy with the Zariski topology) a topology which he calls the verbal topology. He is mainly interested in groups whose verbal topology satisfies the minimal condition on closed sets; for the purposes of this review call such a group a VZ-group. The author proves that various groups are VZ-groups. By far the most surprising result is that every finitely generated abelian-by-nilpotent-by-finite group is a VZ-group. Less surprisingly, every abelian-by-finite group is a VZ-group. So is every linear group. Also, the class of VZ-groups is closed under taking subgroups and finite direct products.

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The Markov topology and the $\mathfrak{P}$-Markov topology

Markov topology $\mathfrak{M}_G$ of G: has as closed sets all unconditionally closed subsets of G

$\mathfrak{M}_G = \inf\{$all Hausdorff group topologies on $G\}$.
(inf taken in the lattice of all topologies on G)
$\mathfrak{P}_G = \inf\{$all precompact group topologies on $G\}$
(G, τ) precompact if the completion is compact Clearly,
$\mathfrak{Z}_G \subseteq \mathfrak{M}_G \subseteq \mathfrak{P}_G$, T_1 topologies

Problem 2. [1944]

Is $\mathfrak{Z}_G = \mathfrak{M}_G$ always true ?

- Perel'man (unpublished): Yes, for abelian groups
- Markov [1944]: Yes, for countable groups.
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Markov's first problem through the looking glass of $\mathfrak{M}_G$

A group G $\mathfrak{Z}$-discrete (resp., $\mathfrak{M}$-discrete, $\mathfrak{P}$-discrete), if $\mathfrak{Z}_G$ (resp., $\mathfrak{M}_G$, resp., $\mathfrak{P}_G$) is discrete. Analogously, define $\mathfrak{Z}$-*compact*, etc.

- G is $\mathfrak{Z}$-discrete if and only if there exist $E_1, \dots, E_n \in \mathfrak{E}_G$ such that $E_1 \cup \ldots \cup E_n = G \setminus \{e_G\}$;
- G is $\mathfrak{M}$-discrete iff G is non-topologizable. So, G is non-topologizable whenever G is $\mathfrak{Z}$-discrete.

Ol′shanskij proved that for Adian group $G = A(n, m)$ the quotient $G/Z(G)^m$ is a countable $\mathfrak{Z}$-discrete group.

Example

(a) Klyachko and Trofimov [2005] constructed a finitely generated torsion-free $\mathfrak{Z}$-discrete group G.

(b) Trofimov [2005] proved that every group H admits an embedding into a $\mathfrak{Z}$-discrete group.

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A group G $\mathfrak{Z}$-discrete (resp., $\mathfrak{M}$-discrete, $\mathfrak{P}$-discrete), if $\mathfrak{Z}_G$ (resp., $\mathfrak{M}_G$, resp., $\mathfrak{P}_G$) is discrete. Analogously, define $\mathfrak{Z}$-*compact*, etc.

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Criterion [Shelah]

An uncountable group G is $\mathfrak{M}_G$-discrete whenever the following two conditions hold:

(a) there exists $m \in \mathbb{N}$ such that $A^m = G$ for every subset A of G with $|A| = |G|$;

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(iii) Using the above criterion, Shelah produced an example of an $\mathfrak{M}$-discrete group under the assumption of CH. Namely, a torsion-free group G of size ω_1 satisfying (a) with $m = 10000$ and (b) with $n = 2$. So G every proper subgroup H of G is malnormal (i.e., $H \cap x^{-1}Hx = \{1\}$), so G is also simple.

Proof.

Let $\mathcal{T}$ be a Hausdorff group topology on G. There exists a $\mathcal{T}$-neighbourhood V of e_G with $V \neq G$. Choose a $\mathcal{T}$-neighbourhood W of e_G with $W^m \subseteq V$. Now $V \neq G$ and (a) yield $|W| < |G|$. Let $H = \langle W \rangle$. Then $|H| = |W| \cdot \omega < |G|$. By (b) the intersection $O = \bigcap_{i=1}^{n} x_i^{-1} H x_i$ is finite for some $n \in \mathbb{N}$ and elements $x_1, \ldots, x_n \in G$. Since each $x_i^{-1} H x_i$ is a $\mathcal{T}$-neighbourhood of 1, this proves that $1 \in O \in \mathcal{T}$. Since $\mathcal{T}$ is Hausdorff, it follows that $\{1\}$ is $\mathcal{T}$-open, and therefore $\mathcal{T}$ is discrete. □

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- *(Bryant) A subgroup of a $\mathfrak{Z}$-Noetherian group is $\mathfrak{Z}$-Noetherian,*
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Using the fact that linear groups are $\mathfrak{Z}$-Noetherian, and the fact that countable free groups are isomorphic to subgroups of linear groups, one gets

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The Zariski topology $\mathfrak{Z}_G$ of the direct product is coarser than the product topology $\prod_{i\in I}\mathfrak{Z}_{G_i}$.

These two topologies need not coincide (for example $\mathfrak{Z}_{\mathbb{Z}\times\mathbb{Z}}$ is the co-finite topology of $\mathbb{Z}\times\mathbb{Z}$, so neither $\mathbb{Z}\times\{0\}$ nor $\{0\}\times\mathbb{Z}$ are Zariski closed in $\mathbb{Z}\times\mathbb{Z}$, whereas they are closed in $\mathfrak{Z}_{\mathbb{Z}}\times\mathfrak{Z}_{\mathbb{Z}}$).

Item (B) of the next theorem generalizes Bryant's result.

Theorem (DD - D. Toller)

(A) Direct products of $\mathfrak{Z}$-compact groups are $\mathfrak{Z}$-compact.
(B) $G=\prod_{i\in I}G_i$ is $\mathfrak{Z}$-Noetherian iff every G_i is $\mathfrak{Z}$-Noetherian and all but finitely many of the groups G_i are abelian.

Corollary

A nilpotent group of nilpotency class 2 need not be $\mathfrak{Z}$-Noetherian.

Take an infinite power of finite nilpotent group, e.g., Q_8.

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If $\{F_i \mid i \in I\}$ is a family of finite groups, and $G = \prod_{i\in I} F_i$, then the product $\prod_{i\in I} \mathfrak{Z}_{F_i}$ is a compact Hausdorff group topology, so

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Theorem (Gaughan 1966)

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$\mathfrak{M}_{Sym(X)}$ coincides with the point-wise convergence topology of $Sym(X)$. Does $\mathfrak{M}_{Sym(X)}$ coincide with $\mathfrak{Z}_{Sym(X)}$?

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If $\{F_i \mid i \in I\}$ is a non-empty family of finite center-free groups, and $G = \prod_{i\in I} F_i$, then $\mathfrak{Z}_G = \mathfrak{M}_G = \mathfrak{P}_G = \prod_{i\in I} \mathfrak{Z}_{F_i}$ is a Hausdorff group topology on G.

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The permutation group $Sym(X)$ of an infinite set X is $\mathfrak{M}$-Hausdorff.

$\mathfrak{M}_{Sym(X)}$ coincides with the point-wise convergence topology of $Sym(X)$. Does $\mathfrak{M}_{Sym(X)}$ coincide with $\mathfrak{Z}_{Sym(X)}$?

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$\mathfrak{P}$-discrete groups

A group G is $\mathfrak{P}$-discrete iff G admits no precompact group topologies (i.e., G is not maximally almost periodic, in terms of von Neumann).

In parfticular, examples of $\mathfrak{P}$-discrete groups are provided by all minimally almost periodic (again in terms of von Neumann, these are the groups G such that every homomorphism to a compact group K is trivial).

Example

(a) (von Neumann and Wiener) $SL_2(\mathbb{R})$;
(b) The permutation group $Sym(X)$ of an infinite set X (as $\mathfrak{M}_{Sym(X)}$ is not precompact).

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Proof.

Let G be a divisible solvable non-abelian group. It suffices to see that G admits no precompact group topology. To this end we show that every divisible precompact solvable group must be abelian.

Let G be a divisible precompact solvable group. Then its completion K is a connected group. On the other hand, K is also solvable. It is enough to prove that G K is abelian.

Arguing for a contradiction, assume that $K \neq Z(K)$, is not abelian. By a theorem of Varopoulos, $K/Z(K)$ is isomorphic to a direct product of simple connected compact Lie groups, in particular, $K/Z(K)$ cannot be solvable. On the other hand, $K/Z(K)$ has to be solvable as a quotient of a solvable group, a contradiction. □

Corollary

For every field K with $\mathrm{char} K = 0$ the Heisenberg group

$$H_K = \begin{pmatrix} 1 & K & K \\ & 1 & K \\ & & 1 \end{pmatrix} \text{ is } \mathfrak{P}\text{-discrete.}$$

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The Zariski topology of an abelian group: Markov's problem 3

Definition (Markov)

A subset A of a group G is potentially dense in G if there exists a Hausdorff group topology $\mathcal{T}$ on G such that A is $\mathcal{T}$-dense in G.

Example (Markov)

Every infinite subset of $\mathbb{Z}$ is potentially dense in $\mathbb{Z}$.
By Weyl's uniform disitribution theorem for every infinite $A = (a_n)$ in $\mathbb{Z}$ there exists $\alpha \in \mathbb{R}$ such that $(a_n\alpha)$ is uniform disitributed modulo 1, so the subset $(a_n\overline{\alpha})$ of $\mathbb{R}/\mathbb{Z}$ is dense in $\mathbb{R}/\mathbb{Z}$ (so in $\overline{\alpha}$ as well). Now the topology $\mathcal{T}$ on $\mathbb{Z}$ induced by $\mathbb{Z} \cong \overline{\alpha} \hookrightarrow \mathbb{R}/\mathbb{Z}$ works.

Problem 3 [Markov]

Characterize the potentially dense subsets of an abelian group.

A hint. [two necessary conditions]

- a potentially dense set is Zarisky-dense;
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If an Abelian group with $|G| \leq \mathfrak{c}$ is either torsion-free or has exponent p, then every infinite set of G is potentially dense.

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Can this be extended to groups with $|G| \leq 2^{\mathfrak{c}}$?

The answer is (more than) positive:

Theorem (DD - D. Shakhmatov)

For a countably infinite subset A of an Abelian group G TFAE:
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Theorem

Let G be an Abelian group with $|G| \leq \mathfrak{c}$ and $\mathcal{E}$ be a countable family in $\mathfrak{T}(G)$. Then there exists a metrizable precompact group topology $\mathcal{T}$ on G such that $cl_{\mathfrak{Z}_G}(S) = cl_{\mathcal{T}}(S)$ for all $S \in \mathcal{E}$.

The realization of the Zariski closure of uncountably many sets is impossible in general.

Corollary

For an abelian group G with $|G| \leq 2^{\mathfrak{c}}$ the following are equivalent:
(a) every infinite subset of G is potentially dense in G;
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$\mathfrak{Z}_G = \mathfrak{M}_G = \mathfrak{P}_G$ *for every abelian group G.*

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For $n \in \omega$ and $E \subseteq G$ let
$G[n] = \{x \in G : nx = 0\}$ and $nE = \{nx : x \in E\}$.
$\forall E \in \mathfrak{E}_G, \exists a \in G, n \in \omega$ such that
$E = a + G[n] = \{x \in G : nx = na\}$.
So $\mathfrak{E}_G$ is stable under finite intersections:
$(a + G[n]) \cap (b + G[m]) = \emptyset$ or $c + G[d]$
with $d = GCD(m, n)$.

Lemma

If G is abelian, then $\mathfrak{A}_G$ consists of finite unions of elementary algebraic sets $\mathfrak{E}_G$. Moreover:
(a) $(G, \mathfrak{Z}_G)$ is Noetherian (hence, compact).
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All these propertirs are false in the non-abelian case (e.g., when G is a countable $\mathfrak{Z}$-discrete group).

Example

$\mathfrak{Z}_G$ coincides with the cofinite topology of an abelian group G iff either $r_p(G) < \infty$ for all primes p or G has a prime exponent p.

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All these propertirs are false in the non-abelian case (e.g., when G is a countable $\mathfrak{Z}$-discrete group).

Example

$\mathfrak{Z}_G$ coincides with the cofinite topology of an abelian group G iff either $r_p(G) < \infty$ for all primes p or G has a prime exponent p.

For $n \in \omega$ and $E \subseteq G$ let
$G[n] = \{x \in G : nx = 0\}$ and $nE = \{nx : x \in E\}$.
$\forall E \in \mathfrak{E}_G, \exists a \in G, n \in \omega$ such that
$E = a + G[n] = \{x \in G : nx = na\}$.
So $\mathfrak{E}_G$ is stable under finite intersections:
$(a + G[n]) \cap (b + G[m]) = \emptyset$ or $c + G[d]$
with $d = GCD(m, n)$.

Lemma

If G is abelian, then $\mathfrak{A}_G$ consists of finite unions of elementary algebraic sets $\mathfrak{E}_G$. Moreover:
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An algebraic description of the $\mathfrak{Z}$-irredducible sets

Definition

A topological space X is irreducible, if $X = F_1 \cup F_2$ with closed F_1, F_2 yields $X = F_1$ or X_2.

Lemma

For a countably infinite subset A of G TFAE:
(a) A is irreducible;
(b) A carries the cofinite tiopology;
(c) there exists $n \in \mathbb{N}$ such that for every $a \in A$

(†) *$E = A - a$ satisfies $nE = 0$ and $\{x \in E : dx = h\}$ is finite for each $h \in H$ and every divisor d of n with $d \neq n$.*

Let $\mathfrak{T}(G) = \{E \in \mathcal{P}(G) : E$ is irreducible and $0 \in cl_{\mathfrak{Z}_G}(E)\}$. For every $E \in \mathfrak{T}(G)$ the set $E_0 = E \cup \{0\}$ is still irreducible. Let $o(E) = o(E_0)$ be the number n determined by (†) and let $\mathfrak{T}_n(G) = \{E \in \mathfrak{T}(G) : o(E) = n\}$. Then $\mathfrak{T}(G) = \bigcup_n \mathfrak{T}_n(G)$, $\mathfrak{T}_1(G) = \emptyset$ and $\mathfrak{T}_m(G) \cap \mathfrak{T}_n(G) = \emptyset$ whenever $n \neq m$.

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$E \in \mathfrak{T}_n(G)$ iff every infinite subset of E is $\mathfrak{Z}_G$-dense in $G[n]$.

Example

Let G be an infinite abelian group.
(a) Every countably infinite subset of G is irreducible if G is torsion-free.
(b) $\mathfrak{T}_0(G) = \emptyset$ iff G is bounded.
(c) $\mathfrak{T}_n(G) = \emptyset$ for some $n > 1$ iff there exists a monomorphism $\bigoplus_\omega \mathbb{Z}(n) \hookrightarrow G$.

Theorem

Let S be an infinite subset of an abelian group G. Then there exist a finite $F \subseteq S$, infinite subsets $\{S_i : i = 1, 2, \ldots, s\}$ of S and a finite set $\{a_1, a_2, \ldots, s\}$ of G such that
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