Uniform conciseness of outer commutator words

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Ischia Group Theory 2010
April 16th, 2010

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So Hall's question amounts to asking: are all words concise?

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- The lower central words $\gamma_i = [x_1, x_2, \ldots, x_i]$.
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- The derived words δ_i, defined recursively by $\delta_0 = x_1$ and

$$\delta_i = [\delta_{i-1}(x_1, \ldots, x_{2^{i-1}}), \delta_{i-1}(x_{2^{i-1}+1}, \ldots, x_{2^i})].$$

  (Turner-Smith, 1964)
  For example, $\delta_1 = [x_1, x_2]$ and $\delta_2 = [[x_1, x_2], [x_3, x_4]]$. The corresponding verbal subgroups are the derived subgroups $G^{(i)}$.

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However, not all words are concise. (Ivanov, 1989)

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Then the ultraproduct U of these groups with respect to a non-principal ultrafilter has at most m values of ω, but $|\omega(U)| = \infty$.

However, neither the ultraproduct argument nor Jeremy Wilson's proof provide an explicit expression for the order of $\omega(G)$ when ω is an outer commutator word.

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- The most important thing in Theorem A is that the bounds are independent of the outer commutator word. This is why we say that outer commutator words are uniformly concise.
- Theorem A does not depend on ultraproducts.
- Our proof of Theorem A is also independent of Wilson's result about the conciseness of outer commutator words.

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Let ω be an outer commutator word, and let G be a soluble group. Then there exists a series of subgroups from 1 to $\omega(G)$ such that:

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Theorem B (F-A, Morigi)

Let ω be an outer commutator word, and let G be a soluble group. Then there exists a series of subgroups from 1 to $\omega(G)$ such that:

- *All subgroups of the series are normal in G.*
- *Every section of the series is abelian and can be generated by values of ω all of whose powers are again values of ω (in the section).*

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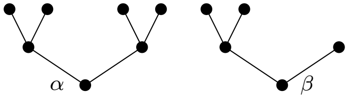

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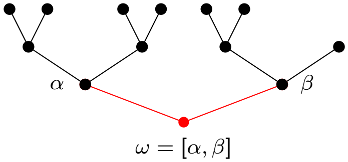

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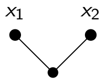

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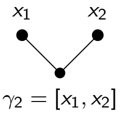

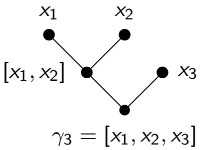

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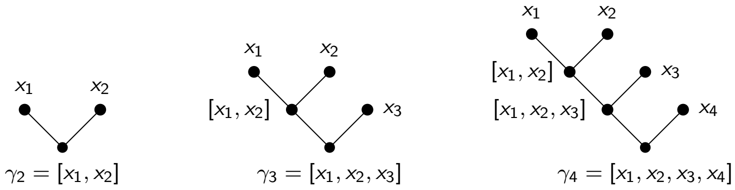

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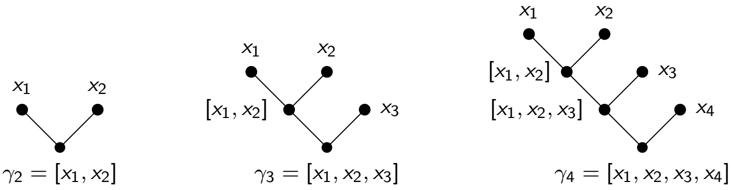

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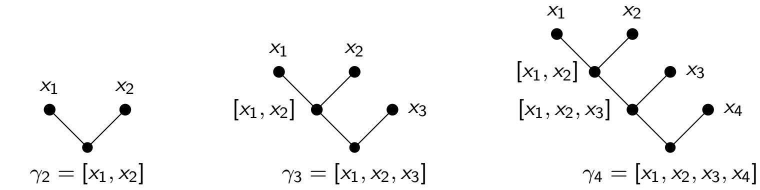

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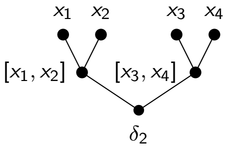

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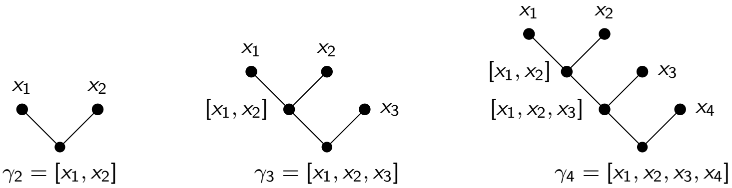

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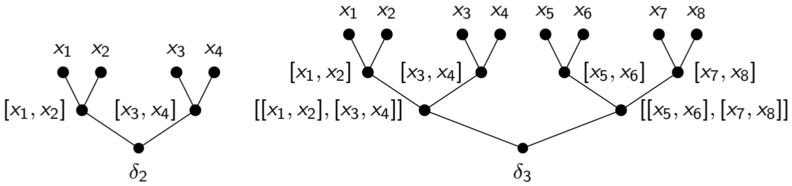

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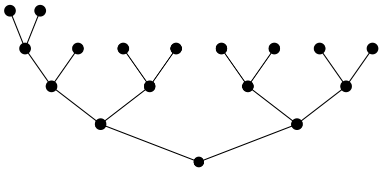

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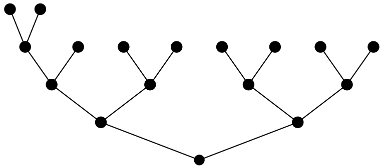

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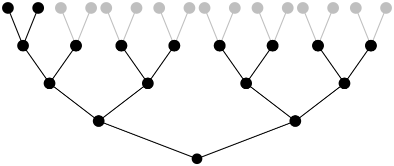

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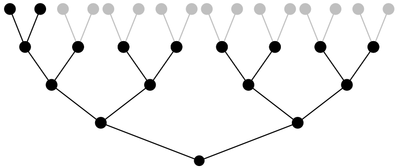

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The word $[[\gamma_3, \gamma_2], \delta_2]$ has height 4 and defect 14.

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- If every value of φ in G is also a value of ω, this is also a PCG-series w.r.t. ω, and we may assume $\varphi(G) = 1$.

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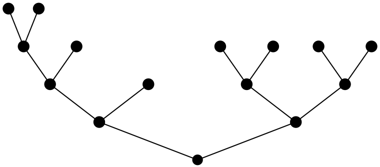

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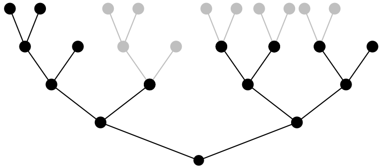

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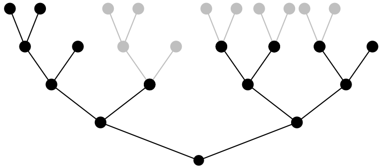

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Making an extension φ of ω corresponds to replacing some indeterminates of ω by other outer commutator words. Hence every value of φ is also a value of ω.

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$$g^n \equiv [a^n, b] \pmod{[\omega(G), \alpha(G)]}.$$

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$$g^n \equiv [a^n, b] \pmod{[\omega(G), \alpha(G)]}.$$
- If $[\omega(G), \alpha(G)]$ is contained in $\prod_{\varphi \in \Phi} \varphi(G)$, then $[\omega(G), \alpha(G)] = 1$, and $g^n = [a^n, b]$, and we have taken the power 'inside the first commutator'.

Taking powers inside ω

- Let Φ be the (finite) set of all outer commutator words of height h which are a proper extension of ω.
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- By applying the induction to α (a word of smaller height), we can take the power inside to the position of an indeterminate, and g^n is a value of ω.

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- Thus, if $\omega = [\alpha, \beta]$ and φ are outer commutator words,
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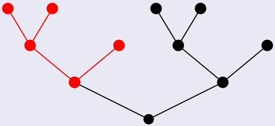
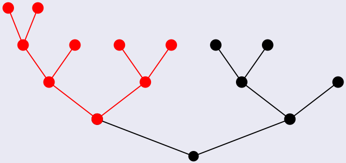


$[\gamma_3, \gamma_3]$



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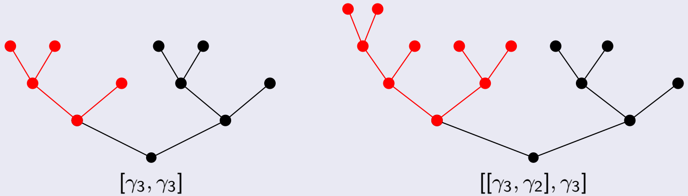

- Thus $\pi^{(1)}$ is obtained from the vertex α and $\pi^{(2)}$ comes from β.

Iterating to sections

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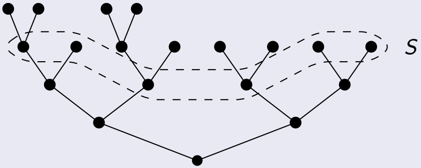
By a section we mean a set of vertices which is obtained when we cut the tree from side to side:



A section S of $[[\gamma_3, \gamma_3], \delta_2]$.

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The existence of S and δ_i is obtained again from the tree of ω.