ISCHIA GROUP THEORY 2010

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ON INFINITE CAMINA GROUPS

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I. **Introduction.**

We start with the basic definition.

Definition. A group G is called a

Camina group

if $G' \neq G$, and for each $x \in G \setminus G'$ the following equation holds:

$$x^G = x\{G'\} \ ,$$

where $x^G = \{x^g \,|\, g \in G\}$ is the **conjugacy class** of x in G and $x\{G'\}$ denotes the **set** $\{xg' \mid g' \in G'\}$.

It is well known that in an arbitrary group

$$x^G \subseteq x\{G'\} \text{ for each } x \in G \ .$$

In Camina groups the **equality**

$$x^G = x\{G'\}$$

holds for each

$$x \in G \setminus G' \ .$$

The class of finite Camina groups was introduced by Alan Camina in 1978 and it had been extensively studied and used since then, in particular by I. D. MacDonald, D. Chillag, C. M. Scoppola, A. Mann, I. M. Isaacs and R. Dark.

Most of these authors considered a more general situation, the so-called "Camina-Pairs".

As far as we know,

infinite Camina groups,

which are the topic of this lecture, had not been investigated before.

II. **Finite Camina groups.**

Camina proved in his 1978 paper that if G is a finite Camina group, then one of the following holds:

- G is a Frobenius group with an abelian complement;
- G' is a p-group for some prime p;
- G/G' is a p-group for some prime p.

In 1984, Chillag and MacDonald proved that if G is a finite Camina group and G' is a p-group, then also G/G' is a p-group.

The **complete** description of finite Camina groups was obtained by Dark and Scoppola in 1996:

Theorem 1. *A finite nonabelian group G is a Camina group if and only if one of the following holds.*

- *G is a Camina p-group of nilpotence class 2 or 3.*
- *G is a Frobenius group with cyclic complements.*
- *G is a Frobenius group with complements isomorphic to the quaternion group of order 8.*

In particular, finite Camina groups are **solvable**, *of Fitting length at most 2.*

III. **Examples of infinite Camina groups.**

There are many examples of infinite nonabelian Camina groups.

They can be much more complicated than the comparatively simple finite Camina groups.
In particular, as will be shown below, there exist **nonsolvable infinite Camina groups**.

We are grateful to Professor A. Ol'shanskii for the nonsolvable examples.

We continue with **examples** of families of infinite Camina groups.

Example 1.

• *Infinite groups G satisfying $|G'| = p$, a prime and $G' = Z(G)$.*

Indeed, if

$$x \in G \setminus G' = G \setminus Z(G),$$

then

$$\{x\} \subset x^G \subseteq x\{G'\} \quad \text{and} \quad |x^G| \text{ divides } |x\{G'\}| = p.$$

Hence

$$x^G = x\{G'\} \quad \text{for each} \quad x \in G \setminus G' \ .$$

Thus these groups are infinite Camina groups.

They are of course nilpotent groups of class 2.

In particular, the infinite extraspecial p-groups are Camina groups.

Example 2.

- *Semidirect products*

$$G = A\langle x\rangle,$$

where

$$A^2 = A \quad \textit{is an infinite group,} \quad |x| = 2$$

and

$$a^x = a^{-1} \quad \textit{for each} \quad a \in A\ .$$

Indeed, A is abelian, $G' \leq A$ and

$$[a, x] = a^{-1}a^x = a^{-2} \quad \text{for each} \quad a \in A\ .$$

Since $A^2 = A$, each $b \in A$ satisfies

$$b = a^{-2} = [a, x] \quad \text{for some} \quad a \in A\ .$$

This implies that $A \leq G'$, yielding

$$G' = A \quad \text{and} \quad G \setminus G' = Ax\ .$$

In order to show that G is a Camina group, we must show that

$$x\{A\} = x\{G'\} = x^G \ .$$

Let $a \in A$. Since $A = A^2$, it follows that

$$a = u^2 \quad \text{for some} \quad u \in A$$

and hence

$$xa = xu^2 = (xu)u = u^{-1}xu = x^u \ .$$

Thus $x\{A\} \subseteq x^G$, yielding our claim.

If A is the Prüfer group of type 2^∞, then G is an infinite Camina group which is solvable, but it is **neither nilpotent nor Frobenius**.

Let p denote a prime. An infinite abelian group is called a *Prüfer group of type* p^∞, if it is isomorphic to the p-primary component of $\mathbb{Q}/\mathbb{Z}$.

Example 3.

• *It is possible to* **modify** *the construction given by Ol'shanskii in his book*

"Geometry of Defining Relations in Groups"

of infinite groups of a finite prime exponent p *(where* p *is big enough), in which* **all subgroups of order** p **are conjugate**,

and to **construct**

infinite nonsolvable groups G

of exponent p **with** $G' \neq G$,

such that

all subgroups of order p **not contained in** G' **are conjugate.**

These groups are Camina groups.

In order to prove that such G is a Camina group, we must show that if $x \in G \setminus G'$ and $a \in G'$, then

$$xa = x^g \quad \text{for some} \quad g \in G\ .$$

Since $x, xa \in G \setminus G'$, if follows by the properties of G that

$$\langle xa \rangle = \langle x \rangle^g \quad \text{for some} \quad g \in G\ .$$

Thus

$$xa = (x^g)^n \quad \text{for some integer} \quad n$$

and it follows from

$$xG' = (xa)G' = (x^g)^n G' = (x[x,g])^n G' = x^n G'$$

and from $|x| = p$ that $n \equiv 1 \pmod{p}$.

Hence

$$xa = (x^g)^n = x^g,$$

as required.

As mentioned above, we are grateful to Professor Ol'shanskii for this important information.

IV. **Preliminary results.**

First we mention a basic lemma.

Lemma 2. *Let G be a nonabelian Camina group.*

Then the following statements hold.

(1) *If $N \trianglelefteq G$ and $N \leq G'$, then also G/N is a Camina group.*

(2) *$Z(G) \leq G'$ and $G/Z(G)$ is a Camina group.*

(3) *Either G is nilpotent or $Z_\infty(G) \leq G'$ and $G/Z_\infty(G)$ is a Camina group.*

Proof of (2). Suppose that there exists

$$x \in Z(G) \setminus G' \ .$$

Since G is a Camina group, we have

$$x^G = x\{G'\} \ .$$

But then

$$1 = |x^G| = |G'|$$

and G is abelian, a contradiction.

Hence $Z(G) \leq G'$ and by (1) $G/Z(G)$ is a Camina group. $\square$

The next corollary deals with the influence of the **structure of** $G/Z(G)$ on the structure of a nonabelian Camina group G.

Corollary 3. *Let G be a nonabelian Camina group. Then the following statements hold.*

(1) *If $G/Z(G)$ is finite, then also G is finite.*

(2) *If $G/Z(G)$ is a locally finite π-group, where π is a set of primes, then also G is a locally finite π-group.*

(3) *If $G/Z(G)$ is a Černikov group, then also G is a Černikov group.*

A group is called a Černikov group if it is an extension of a finite direct product of Prüfer groups by a finite group.

Proof of (2). Since $G/Z(G)$ is a locally finite π-group, it follows by the Corollary on page 102 of Derek Robinson's book "Finiteness Conditions and Generalized Soluble Groups", part 1, that also G' is a locally finite π-group. Since, by Lemma 2(2), $Z(G) \leq G'$, it follows that $Z(G)$, and hence also G, are locally finite π-groups. $\square$

We conclude this section with results concerning Camina groups with a **finite commutator subgroup**.

We begin with the following basic theorem.

Theorem 4. *Let G be a nonabelian group with G' finite. Then G is a Camina group if and only if one of the following holds.*

(1) *G is a finite Camina group.*

(2) *G is an infinite nilpotent p-group of class 2 and of exponent p or p^2, with $Z(G) = G'$, and for any maximal subgroup H of G', G/H is an extraspecial group.*

Theorem 4 yields the following corollary.

Corollary 5. *Let G be a nonabelian finitely generated Camina group with G' finite. Then G is a finite group.*

Proof. Supose that G is an infinite group. Then, by Theorem 4, G is a finitely generated torsion nilpotent group, hence finite, a contradiction. $\square$

V. **Main results.**

First we consider **locally finite** Camina groups.

Theorem 6. *Let G be a nonabelian locally finite Camina group. Then one of the following holds.*

(1) *G/G' is a p-group, for a suitable prime p.*

(2) *G' is a nilpotent group, and either G' is a p-group (p a prime), or G/G' is a locally cyclic group.*

Next we deal with **residually finite** Camina groups.

Theorem 7. *Let G be a nonabelian residually finite Camina group. Then one of the following holds.*

(1) *G is a finite p-group of nilpotency class ≤ 3.*

(2) *G is an infinite nilpotent p-group of class 2 and of exponent p or p^2, with $G' = Z(G)$.*

(3) *G is a Frobenius group with the Frobenius kernel G' of nilpotency class h, where h is a number depending only on $|G/G'|$, and with finite cyclic Frobenius complements.*

(4) *G is a Frobenius group with an abelian Frobenius kernel and with Frobenius complements isomorphic to the quaternion group Q_8.*

In any case G is solvable.

Theorem 7 yields the following corollary.

Corollary 8. *Let G be a nonabelian polycyclic-by-finite Camina group. Then G/G' is finite.*

Proof. Clearly G is **finitely generated**. Moreover, G being polycyclic-by-finite implies, by an extension of a result of Hirsh (see Exercises on page 157 of D. Robinson's book "A Course in the Group Theory"), that G is also **residually finite**. Hence the corollary follows by Theorem 7, since case (2) of that theorem is impossible, in view of G being finitely generated. $\square$

We now state our **major result** concerning **finitely generated** infinite Camina groups.

Theorem 9. *Let G be a nonabelian finitely generated infinite Camina group. Then G is nonsolvable.*

Equivalently, Theorem 9 implies that:

A nonabelian finitely generated **solvable** *Camina group if* **finite**.

The proof of Theorem 9 is quite complicated. We shall mention here only one preliminary result.

The following proposition served as a step in the proof of Theorem 9.

Proposition 10. *Let G be a nonabelian finitely generated solvable Camina group.*
Then G/G' is finite.

Proof. If G is nilpotent, then G is polycyclic and G/G' is finite by Corollary 8. So we may assume that G is non-nilpotent. Then by Theorem 10.51 of Robinson's book "Finiteness Conditions and Generalized Soluble Groups", part 1, there exists a normal subgroup N of G such that G/N **is finite and non-nilpotent**.

Thus $N \cap G' < G'$ and by Lemma 2(1),

$G/(N \cap G')$ is a

finitely generated non-abelian Camina group

with the commutator subgroup

$$(G/(N \cap G'))' = G'/(N \cap G') \simeq G'N/N$$

which is isomorphic to a subgroup of G/N, and

hence **finite**. Consequently, by Corollary 5,

$G/(N \cap G')$ is finite, which implies that

G/G' is finite, as required. $\square$

Theorem 9 yields the following corollary.

Corollary 11. *Let G be a finitely generated nonabelian Camina group. Then the following statements hold.*

(1) *If G is residually finite, then it is finite.*

(2) *If G is a linear group, then it is finite.*

Proof of (1). Since G is residually finite, it follows by Theorem 7 that G is solvable and hence it is finite by Theorem 9. $\square$

Our last topic in this talk are **periodic solvable Camina groups** G, with an infinite commutator subgroup G' satisfying one of the following condition:

1. G' satisfies the *min*-condition.

2. G' is of a finite Prüfer rank.

We start with the appropriate definitions.

Definitions.

(1) We say that a group G satisfies the *min* condition if G satisfies *the minimal condition on subgroups.*

(2) We say that a group G has a *finite Prüfer rank* r if every finitely generated subgroup of G can be generated by r elements and r is the least such integer.

We also mention the following fact.

Proposition 12. *A solvable group has a finite Prüfer rank if and only if it has a series of finite length whose factors are either infinite cyclic or isomorphic to subgroups of* $\mathbb{Q}/\mathbb{Z}$.

We obtained the following results concerning periodic solvable Camina groups G, with an infinite commutator subgroup G'.

Theorem 13. *Let G be a periodic solvable Camina group and suppose that G' is an infinite group. Then the following statements hold.*

(1) *If G' satisfies the min condition, then G/G' is finite.*

(2) *If G' is of finite Prüfer rank, then G/G' is finite.*

In addition to Theorem 13, we also obtained a complete characterization of the corresponding groups.

We shall describe here only one of these characterizations.

Theorem 21. *Let G be a periodic solvable group and assume that G' is an infinite group, of finite Prüfer rank. If G is a Camina group, then G/G' is finite and one of the following holds.*

(1) *G is a Camina p-group satisfying the min condition.*

(2) *G is a Frobenius group, with complements which are either cyclic or isomorphic to the quaternion group Q_8.*

(3) $G = A \rtimes K$, *where* K *is a finite abelian group,* A *is direct product of finitely many Prüfer* p_i*-groups (where* p_i *are primes, not necessarily distinct), and* $C_A(k)$ *is finite for any* $k \in K \setminus \{1\}$.

(4) $G = (B \times C) \rtimes K$, *where* K *is a finite abelian group,* $B \times C$ *is a nilpotent group,* $C \rtimes K$ *is a Frobenius group with the kernel* C, B *is a direct product of finitely many Prüfer* p_i*-groups (*p_i *primes), and* $C_B(k)$ *is finite for any* $k \in K \setminus \{1\}$.

(5) $G = (B \times C)K$, *where* $B \times C$ *is normal in* G, K *is a finite p-group (p a prime),* C *is a* p'*-group,* B *is the direct product of finitely many Prüfer p-groups,* G/B *is a Frobenius group with the kernel* $(B \times C)/B$ *and with complements which are either cyclic or isomorphic to the quaternion group, and* $C_B(y)$ *is finite for any* $y \in K \setminus B$.

Conversely, if (1), (2), (3), (4) *or (5) holds, then* G *is a Camina group.*

THE END