Twisted conjugacy in certain Artin Groups and Coxeter Groups

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Dedicated to the memory of Karl Gruenberg

(1.1) Decision Problems

Let G be a group. Two of the basic decision problems are the

(i) *Word Problem* (W.P.): given elements $g_1,\ g_2 \in G$, decide whether $g_1 = g_2$ in G (or $g_1 g_2^{-1} = 1$). Undecidable, in general.

(ii) The *Conjugacy Problem* (C.P.): given elements g_1 and g_2 in G, decide whether an element $h \in G$ exists such that $h^{-1} g_1 h = g_2$. The element h is a *conjugator*.

(1.2) Decision Problems in embedded subgroups

Let K_1 be a group, α an embedding of K_1 into G. Let $K = \alpha(K_1)$. If G has solvable W.P. then clearly K has. However, it is possible that G has solvable C.P. but K has unsolvable C.P. (Example due to Don Collins and Charles Miller, Proc. London. 34(3)).

Definition

α is a *Frattini embedding* if for every $u, v \in K$ they are conjugate by an element of G if and only if they are conjugate by an element of K. α is a Frattini embedding $\Longrightarrow$ if G has solvable C.P. then K_1 has a solvable C.P.

(1.3) Twisted Conjugacy Problem (T.C.P.)

Let $\sigma \in Aut(G)$. Let $a, b \in G$.

(i) T.C.P.: Decide whether an element $h \in G$ exists such that $h^{-1}a(h)^{\sigma} = b$. We call h a *σ-conjugator of a to b.*

(ii) Let K be a σ-invariant subgroup of G. K is *σ-Frattini embedded in G* if for every $u, v \in K$, they are σ-conjugate by an element of G if and only if they are σ-conjugate by an element of K.

(1.4) Artin Groups

(i) *Standard Presentation of G.* $< x_1, \ldots, x_n | R_{ij}, \ \ 1 \leq i < j \leq n >$, where either $R_{ij} = 1$ or $|R_{ij}| = 2n_{ij}$ and $R_{ij} : U = V$, where U is an initial subword of $(x_i x_j)^{n_{ij}}$ and V is an initial subword of $(x_j x_i)^{n_{ij}}$ of length n_{ij}.

(ii) *Associated graph* Γ: vertices $x_1, \ldots, x_n$, if $R_{ij} \neq 1$ then x_i and x_j are connected by an edge labeled n_{ij}.

The main result

Our main result relies heavily on the precise description of the automorphism group of the class of Artin groups with which we deal, due to John Crisp: "Automorphisms and abstract commen-surators of 2-dimensional Artin groups." Geometry and Topology 9 (2005) 1381-1441. The Main Result is the following

Theorem *Let G be an Artin group generated by at least 3 generators. Assume that Γ_G satisfies each of the following:*

1) Γ_G *is connected*;

2) $m_{ij} \geq 3$;

3) Γ_G *has no triangles.*

Let $\sigma \in Aut(G)$. *Then each of the following holds*:

a) *G has solvable TCP*;

b) *Let $u, v \in G$ be presented by reduced words U and V, respectively. If u and v are σ-conjugate then there exists a σ-conjugator h presented by a reduced word H, such that*

$|H| \leq max\{|R||U|+|\sigma|,\ |R||V|+|\sigma|\}$, where R is the longest relator and $|\sigma| = max\{\sigma(x)|x = x_1, \ldots, x_n\}$.

c) *G has infinite number of σ-conjugate classes*;

d) *Let P be a σ-invariant parabolic subgroup of G. Then P is σ-Frattini embedded in G;*

e) *Let S be the submonoid of G generated by $x_1, \ldots, x_n$. If S is σ-invariant then S is weakly Frattini embedded in G, in the sense that if $u, v \in S$ and u and v are σ-conjugate in G then there exists an element $s \in S$, presented by a reduced positive word T with $|T| \leq max\{|R||U| + |\sigma|,\ |R||V| + |\sigma|\}$, such that either $sv = u(s)^\sigma$ or $su = v(s)^\sigma$.*

The proof is by a careful analysis of annular cancellation diagrams of the extension $G\sigma =< G, t|t^{-1}x_it = x_i^{\sigma_i} = 1, \ldots, n >$ of G.