Group-theoretic applications of Lie rings with finite cyclic grading

Evgeny KHUKHRO

Idea:

How results about Lie rings with finite cyclic grading,

even rather “extravagant”,

find applications to groups and Lie algebras.

Lie ring method:

hypothesis on a group $\qquad$ hypothesis on a Lie ring

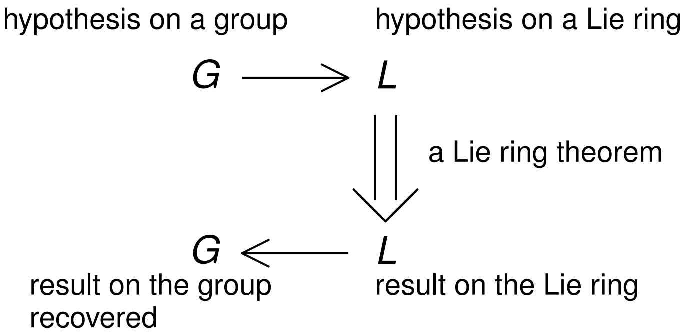
$$G \longrightarrow L$$

$$\Big\Downarrow \quad \text{a Lie ring theorem}$$

$$G \longleftarrow L$$

result on the group recovered $\qquad$ result on the Lie ring

Lie ring method:



1. A hypothesis on a group is translated into a hypothesis on a Lie ring constructed from the group in some way.

2. Then a theorem on Lie rings is proved (or used).

3. Finally, a result about the group must be recovered from the Lie ring information obtained.

Various Lie ring methods:

1. For complex and real Lie groups: Baker–Campbell–Hausdorff formula, EXP and LOG functors
2. Mal'cev's correspondence based on Baker–Campbell–Hausdorff formula for torsion-free (locally) nilpotent groups
3. Lazard's correspondence (including for p-groups of nilp. class $< p$)
4. Lie rings associated with uniformly powerful p-groups

   But most "democratic", for any group:

5. Associated Lie ring

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Definition: associated Lie ring $L(G)$

For any group G: $$L(G) = \bigoplus_i \gamma_i(G)/\gamma_{i+1}(G)$$

Lie product for homogeneous elements:
$$[a + \gamma_{i+1},\ b + \gamma_{j+1}]_{\text{Lie ring}} \ = \ [a, b]_{\text{group}} + \gamma_{i+j+1}$$

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Minuses: Only about $G/\bigcap \gamma_i(G)$,
so only for (residually) nilpotent groups.
Even for these, some information may be lost:
e. g., derived length may become smaller.

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$L = L_0 \oplus L_1 \oplus \cdots \oplus L_{n-1}$

L_i additive subgroups satisfying $[L_i, L_j] \subseteq L_{i+j \,(\mathrm{mod}\, n)}$.

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Higman 57, Kostrikin–Kreknin 63, Kreknin 63

If $L_0 = 0$, then L is soluble of derived length $\leqslant k(n)$;

If in addition $n = p$ is a prime, then L is nilpotent of class $\leqslant h(p)$.

Estimates for Kreknin's and Higman's functions

$k(n) \leqslant 2^n - 2$

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Confirmed for $p = 3, 5, 7, 11$

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Higmans's examples $h(p) \geqslant \dfrac{p^2 - 1}{4}$

Fixed-point-free automorphisms

Let a Lie ring M admit an automorphism φ of order n:
after adjoining a primitive nth root of unity ω
we obtain $M = M_0 + M_1 + \cdots + M_{n-1}$,
where $M_i = \{x \in M \mid \varphi(x) = \omega^i x\}$,

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Earlier Engel–Jacobson–Borel–Mostow for finite-dimensional only and without upper bounds.

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(Apart from connected simply connected Lie groups with fixed-point-free automorphism of finite order.)

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Corollary (Higman 57)

If a (locally) nilpotent group G has an automorphism $\varphi \in Aut\, G$ of prime order p such that $C_G(\varphi) = 1$, then G is nilpotent of class $\leqslant h(p)$.

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(Also true for any finite group G, nilpotent by Thompson 59.)

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Here $L = L(G)$ does not work as derived length is not preserved.

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If a locally nilpotent torsion-free group G has an automorphism $\varphi \in \mathrm{Aut}\, G$ of finite order n such that $C_G(\varphi) = 1$, then G is soluble of derived length $\leqslant 2^n - 2$.

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Embed G into its Mal'cev completion $\hat{G}$ by adjoining all roots of nontrivial elements;

then φ extends to $\hat{G}$ with $C_{\hat{G}}(\varphi) = 1$.

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Let L be the Lie algebra over $\mathbb{Q}$ in the Mal'cev correspondence with $\hat{G}$ given by Baker–Campbell–Hausdorff formula.

Then φ can be regarded as an automorphism of L with $C_L(\varphi) = 0$.

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By Kreknin, L is soluble of derived length $2^n - 2$;

hence so is $\hat{G}$, and so is G.

Endimioni 2010

If a polycyclic group G has an automorphism $\varphi \in Aut\ G$ of prime order p with finite fixed-point subgroup, $|C_G(\varphi)| < \infty$, then G has a subgroup of finite index that is nilpotent of class $\leqslant h(p)$.

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Proof: by reduction to finite q-groups with fixed-point-free automorphism φ, to which Higman's theorem is applied.

Remark 2010

If a polycyclic group G has an automorphism $\varphi \in Aut\ G$ of finite order n with finite fixed-point subgroup, $|C_G(\varphi)| < \infty$, then G has a subgroup of finite index that is soluble of derived length $\leqslant k(n) + 1$.

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Now Folklore's theorem above can be applied to $[H, H]$, so H is soluble of derived length $\leqslant k(n) + 1$.

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Proofs use Higman's and Kreknin's theorems.

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Proof based on Kreknin's theorem.

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Immediate corollaries, with the same conclusions, for Lie rings with automorphism $\varphi^n = 1$ and $\dim C_L(\varphi) = r$ (or $|C_L(\varphi)| = r$).

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Non-trivial even for finite-dimensional, because of those bounds.

Group-theoretic corollaries

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Proof is based on the Lie ring method. But difficult recovery $G \longleftarrow L$, as there is no good correspondence for subgroups$\leftrightarrow$subrings.

Group-theoretic corollaries for rank of fixed points

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If a finitely generated, or periodic, nilpotent group G admits an automorphism φ of prime order p with centralizer $C_G(\varphi)$ of finite rank r, then G has a characteristic nilpotent subgroup C of p-bounded class such that G/C has finite (p, r)-bounded rank.

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Proof is based on the Lie ring method.

But difficult even $G \longrightarrow L$, since rank can increase.

Also difficult recovery $G \longleftarrow L$, as there is no good correspondence for subgroups$\leftrightarrow$subrings.

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Suppose that a finite group G of rank r admits an automorphism φ of order n with $|C_G(\varphi)| = m$. Then G has a characteristic soluble (if n is a prime, nilpotent) subgroup C of r-bounded derived length (nilpotency class) such that G/C has finite (n, m, r)-bounded rank.

Few non-zero components, continued

Makarenko 07, Khukhro–Makarenko–Shumyatsky 08

Suppose that in $L = L_0 \oplus \cdots \oplus L_{n-1}$ there are only d non-zero components among the L_i and $\dim L_0 = r$ (or $|L_0| = r$). Then L contains a soluble (for n prime, nilpotent) ideal of d-bounded derived length (class) and of (d, r)-bounded codimension.

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Applications to Lie algebras (and therefore certain groups):

generalization of Jacobson's theorem on nilpotent algebras of derivations to the case of “almost without nontrivial constants”;

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Applications to Lie algebras (and therefore certain groups):

generalization of Jacobson's theorem on nilpotent algebras of derivations to the case of "almost without nontrivial constants";

to Lie algebras with almost fixed-point-free automorphisms that have few eigenvalues.

$(\mathbb{Z}/n\mathbb{Z})$-graded Lie ring L with many commuting components

Let L be a $(\mathbb{Z}/n\mathbb{Z})$-graded Lie ring.

Khukhro 09

Suppose that for some m for we have

$$|\{i \mid [L_k, L_i] \neq 0\}| \leqslant m,$$

i. e. each component L_k commutes with all but at most m components. If $L_0 = 0$, then L is soluble (for n prime, nilpotent) of m-bounded derived length (class).

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Found application to 2-Frobenius groups.

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Mazurov's problem:

Is the nilpotency class of G bounded in terms of $|A|$ and the nilpotency class of $C_G(A)$?

2-Frobenius groups continued

The above theorem on many commuting components was applied to the case $C_G(A)$ abelian.

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Khukhro 09

Let GBA be a 2-Frobenius group. If $C_G(A)$ is abelian, then G is nilpotent of $|A|$-bounded class.

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Recall that G is nilpotent of $|B|$-bounded class. This theorem gives a better bound if $|A|$ is small. Earlier proved by Mazurov for $|A| = 2$ and 3.

2-Frobenius groups continued

Most recent progress was made by Makarenko–Shumyatsky 2010 in replacing "$C_G(A)$ abelian" by "$C_G(A)$ nilpotent of class c" resulting in the affirmative solution of Mazurov's problem: G is nilpotent of $(c, |A|)$-bounded class. (Talk of Natasha Makarenko.)

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Underlying Lie ring result is a generalization of the theorem on many commuting components above.

Remark on characteristic subgroups

Wherever a subgroup of (bounded) finite index s is constructed that is nilpotent of class t, or soluble of derived length t, there is also a characteristic subgroup of (s, t)-bounded index which is nilpotent of class $\leqslant t$, or soluble of derived length $\leqslant t$, respectively.

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The same is true for a normal subgroup H of a p-group G or of a torsion-free group with G/H of (bounded) rank r such that H satisfies some multilinear commutator law.

Moreover, Khukhro–Klyachko–Makarenko–Mel'nikova 09: similar results for subgroups or ideals H satisfying multilinear commutator laws, with G/H not only finite, or of finite rank, or of dimension, but certain general 'smallness' property satisfying certain axioms (for general groups with multioperators).