Groups and Lie rings with Frobenius groups of automorphisms

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- Part (a). Can the nilpotency class of G be bounded in terms of $|H|$ and the class of $C_G(H)$?
- Part (b). Can the exponent of G be bounded in terms of $|H|$ and the exponent of $C_G(H)$?

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Thus, F is cyclic.

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If $C_G(F) = 1$ and $C_G(H)$ is nilpotent of class c, then the nilpotency class of G is bounded in terms of $|H|$ and c.

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In general case if we do not assume that $(|G|, |H|) = 1$, this equality may no longer be true.

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Let a finite group G admit a Frobenius group of automorphisms FH with cyclic kernel F of order n and complement H of order q. Suppose that $C_G(F) = 1$, $C_G(H)$ is nilpotent of class c.

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If n has no divisors $\leq f(c, q)$, then the nilpotency class of G is bounded in terms of q and c.

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where n is the nilpotency class of G and the γ_i are the terms of the lower central series of G.

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where n is the nilpotency class of G and the γ_i are the terms of the lower central series of G.

The nilpotency class of G coincides with that of $L(G)$.

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Particular case. F is cyclic of prime order p. G admits a Frobenius group of automorphisms FH with kernel F and complement H.

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It can be proved that $C_L(F) = 0$ and $C_L(H)$ is nilpotent of class c.

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The group FH acts in a natural way on $\tilde{L}$ and the action satisfies the conditions that $C_{\tilde{L}}(F) = 0$ and $C_{\tilde{L}}(H)$ is nilpotent of class c.

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Thus, it is sufficient to show that a $\mathbb{Z}/p\mathbb{Z}$-graded Lie ring L with $L_0 = 0$ and nilpotent $C_L(H)$ of nilpotency class c is nilpotent of (c, q)-bounded class.

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Step 2: Combinatorial condition

Definition.

Let q, p, r be positive integers such that p is prime, q divides $p-1$, $1 \leq r \leq p-1$ and r is primitive qth root of 1 in $\mathbb{F}_p$. Let $a_1, \dots, a_k$ be not necessarily distinct elements of $\mathbb{F}_p$. We say that the sequence $(a_1, \dots, a_k)$ is *r-dependent* if and only if there exist $i_1, \dots, i_m \in \{1, 2, \dots, k\}$ and $\alpha_1, \dots, \alpha_m \in \{1, 2, \dots, q-1\}$ such that

$$a_{i_1} + \cdots + a_{i_m} = r^{\alpha_1} a_{i_1} + \cdots + r^{\alpha_m} a_{i_m}.$$

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Example: $q = 2$, $p = 13$, $r = -1$. We take, for example, $(1, 1, 2, 3)$.

The sequence $(1, 1, 2, 3)$ is r-independent, because we cannot find $i_1, \ldots, i_m \in \{1, 2, 3, 4\}$ such that

$$a_{i_1} + \cdots + a_{i_m} = -a_{i_1} + \cdots - a_{i_m} (\mathrm{mod}\, 13).$$

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Proposition.

Let q, p, r, c be positive integers such that p is prime, q divides $p-1$, $1 \leq r \leq p-1$ and r is primitive qth root of 1 in $\mathbb{F}_p$.

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Proposition.

Let q, p, r, c be positive integers such that p is prime, q divides $p-1$, $1 \le r \le p-1$ and r is primitive qth root of 1 in $\mathbb{F}_p$. Let

$$L = \sum_{i=0}^{p-1} L_i$$

be a $\mathbb{Z}/n\mathbb{Z}$-graded Lie ring such that $L_0 = 0$ and

$$[x_{d_1}, x_{d_2}, \ldots, x_{d_{c+1}}] = 0 \text{ whenever } (d_1, \ldots, d_{c+1}) \text{ is } r\text{-independent},$$

where $x_{d_1}, x_{d_2}, \ldots, x_{d_{c+1}}$ are homogeneous elements $x_{d_i} \in L_{d_i}$.

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Then L is nilpotent of (c, q)-bounded derived length.

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By the same reasons we have, for example,

$$[L_1, L_1, L_1, L_1] = 0, [L_2, L_2, L_2, L_2] = 0,$$

$$[L_1, L_2, L_3, L_1] = 0, \text{ etc.}$$

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Step 3: Proof of the Proposition.

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Some ideas of the work of Khukhro on Lie ring with few number of commuting components are also used.