Uniform triples and fixed point spaces

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Joint with Robert M. Guralnick (USC)

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Here, $\dim C_V(g) \geq \frac{1}{3} \dim V$.

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$G \leq \mathrm{GL}(V)$ *finite irreducible solvable* $\Longrightarrow$ *there exists* $g \in G$ *with*

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Uses classification of finite simple groups (CFSG)

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Both use CFSG

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Best possible by example of $\mathrm{SO}_3(k)$.

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Example

$G_m = \mathfrak{A}_5 \times \ldots \times \mathfrak{A}_5$ (m copies) acting on
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$$\Longrightarrow \qquad \dim C_{V_m}(g) > \frac{1}{50} \dim V_m \quad \text{for all } g \in G_m.$$

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- *N non-abelian: Reduces to simple case (for $\mathrm{L}_2(2^f)$ use an argument of Guralnick–Maróti)*

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(since $\dim C_V(g_i) = \dim V - \dim[g_i, V]$)

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Thus, Thm. A follows from certain generation property of simple groups.

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Still, a slight variation holds in this case as well.

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So: $\epsilon(C) < 1 \implies n(C) > 0 \implies$ there exist triples

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Weigel (1992): if $x \in M < G$ then $M \leq N_G(T) = T : Z_{30}$

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C class of $x \in G$ of order $q^8 + q^7 - q^5 - q^4 - q^3 + q + 1$,
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(3) $(n, q) = (10, 2)$, $M = \mathfrak{A}_{12}$;

(4) $(n, q) = (12, 2)$, $M = \mathfrak{A}_{13}, \mathrm{L}_2(13), \mathrm{L}_3(3)$; or

(5) $(n, q) = (18, 2)$, $M = \mathfrak{A}_{20}$.

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$\Longrightarrow$ Thm. C holds for these groups

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G finite non-abelian simple, $G \neq \mathrm{O}_8^+(2) \Longrightarrow$
there exists an element x of order prime to 6 such that:

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Take x as in Thm. C.
If contained in at most two maximal subgroups done by: □

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Lemma

C a non-trivial conjugacy class in a finite simple group G
$\Longrightarrow C$ not contained in the union of any two proper subgroups.

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or: use fixed point ratios