Average dimension of fixed point spaces with applications

Attila Maróti

(Rényi Institute of Mathematics, Budapest, Hungary)

Ischia, 14th of April, 2010

I will talk about a joint work with Robert M. Guralnick.

A conjecture of Neumann and Vaughan-Lee

Let G be a finite group, F a field, and V a finite dimensional FG-module. For a non-empty subset S of G we define

$$\mathrm{avgdim}(S, V) = \frac{1}{|S|} \sum_{s \in S} \dim C_V(s)$$

to be the arithmetic average dimension of the fixed point spaces of all elements of S on V.
In his 1966 DPhil thesis Peter M. Neumann conjectured that if V is a non-trivial irreducible FG-module then

$$\mathrm{avgdim}(G, V) \leq (1/2) \dim V.$$

This problem was restated in 1977 by Neumann and Vaughan-Lee and was posted in 1982 by Vaughan-Lee in The Kourovka Notebook as Problem 8.5.

Known results

The following results were known about the conjecture of Neumann and Vaughan-Lee.

- true if $|G|$ is invertible in F (Neumann, Vaughan-Lee);
- true if G is solvable (Neumann, Vaughan-Lee);
- $\mathrm{avgdim}(G, V) \leq (3/4) \dim V$ (Segal, Shalev);
- $\mathrm{avgdim}(G, V) \leq ((p+1)/2p) \dim V$ where p is the smallest prime divisor of $|G|$ (Isaacs, Keller, Meierfrankenfeld, Moretó).

The solution of the conjecture

Theorem 1

Let G be a finite group, F a field, and V a finite dimensional FG-module. Let N be a normal subgroup of G that has no trivial composition factor on V. Then $\mathrm{avgdim}(Ng, V) \leq (1/p) \dim V$ for every $g \in G$ where p is the smallest prime factor of the order of G.

Some remarks about Theorem 1.

- The group G need not be irreducible on V; the only restriction imposed is that G has no trivial composition factor on V.
- We prove the bound $(1/2) \dim V$ not just for $\mathrm{avgdim}(G, V)$ but for $\mathrm{avgdim}(S, V)$ where S is an arbitrary coset of a normal subgroup of G with a certain property.
- Theorem 1 involves a better general bound, namely $(1/p) \dim V$ where p is the smallest prime divisor of $|G|$. This bound is best possible in the sense that equality is attained in infinitely many cases.

Ingredients

There are two new main ingredients in the proof of Theorem 1. The first is a consequence of the proof of the main result of the paper by Breuer, Guralnick, Kantor (2008).

Lemma

Let G be a finite group with a minimal normal subgroup $N = L_1 \times \ldots \times L_t$ for some positive integer t with $L_i \cong L$ for all i with $1 \leq i \leq t$ for a non-abelian simple group L. Assume that $G/N = \langle xN \rangle$ for some $x \in G$. Then there exists an element $s \in L_1 \leq N$ such that $|\{g \in Nx : G = \langle g, s \rangle\}| > (1/2)|N|$.

Scott's Lemma, 1977

Let G be a subgroup of $GL(V)$ with V a finite dimensional vector space. Suppose that $G = \langle g_1, \ldots, g_r \rangle$ with $g_1 \cdots g_r = 1$. Then

$$\sum_{i=1}^{r} \dim[g_i, V] \geq \dim V + \dim[G, V] - \dim C_V(G).$$

Choosing an element with small fixed point space

In his DPhil thesis Peter M. Neumann showed that if V is a non-trivial irreducible FG-module for a field F and a finite solvable group G then there exists an element of G with small fixed point space. Similar results were obtained for arbitrary groups G by Segal and Shalev, and later, by Isaacs, Keller, Meierfrankenfeld, and Moretó. A direct consequence of Theorem 1 (obtained by noting that $\dim C_V(1) = \dim V$) is the following.

Corollary 2

Let G be a finite group, F a field, and V a finite dimensional FG-module. Let N be a normal subgroup of G that has no trivial composition factor on V. Let x be an element of G and let p be the smallest prime factor of the order of G. Then there exists an element $g \in Nx$ with $\dim C_V(g) \leq (1/p) \dim V$ and there exists an element $g \in N$ with $\dim C_V(g) < (1/p) \dim V$.

Under this generality, the bounds in Corollary 2 are best possible.

A conjecture of Isaacs, Keller, Meierfrankenfeld, and Moretó

Let $\mathrm{cl}_G(g)$ denote the conjugacy class of an element g in a finite group G, and for a positive integer n and a prime p let n_p denote the p-part of n. Isaacs, Keller, Meierfrankenfeld, and Moretó conjectured that for any primitive complex irreducible character χ of a finite group G the degree of χ divides $|\mathrm{cl}_G(g)|$ for some element g of G. We prove

Corollary 3

Let χ be an arbitrary primitive complex irreducible character of a finite solvable group G and let p be a prime divisor of $|G|$. Then the number of $g \in G$ for which $\chi(1)_p$ divides $(|\mathrm{cl}_G(g)|)^3$ is at least $(2|G|)/(1+k)$ where $k = \log_p |G|_p$. Furthermore if $\chi(1)_p > 1$ then there exists a p'-element $g \in G$ for which $p^3 \cdot \chi(1)_p$ divides $(|\mathrm{cl}_G(g)|)^3$.

Non-abelian version of Theorem 1

Let G be a finite group acting on another finite group Z by conjugation. For a non-empty subset S of G define

$$\mathrm{geom}(S, Z) = \left(\prod_{s \in S} |C_Z(s)| \right)^{1/|S|}$$

to be the geometric mean of the sizes of the centralizers of elements of S acting on Z. Similarly, for a non-empty subset S of G define

$$\mathrm{avg}(S, Z) = \frac{1}{|S|} \sum_{s \in S} |C_Z(s)|$$

to be the arithmetic mean of the sizes of the centralizers of elements of S acting on Z.

Theorem 4

Let G be a finite group with $X/Y = M$ a non-abelian chief factor of G with X and Y normal subgroups in G. Then, for any $g \in G$, we find that $\mathrm{geom}(Xg, M) \leq \mathrm{avg}(Xg, M) \leq |M|^{1/2}$.

A consequence of Theorems 1 and 4

A chief factor X/Y of a finite group G is called central if G acts trivially on X/Y and non-central otherwise. Let $\mathrm{ccf}(G)$ and $\mathrm{ncf}(G)$ be the product of the orders of all central and non-central chief factors (respectively) of a finite group G. (In case these are not defined put them equal to 1.) These invariants are independent of the choice of the chief series of G. Let $F(G)$ denote the Fitting subgroup of G. Note that $F(G)$ acts trivially on every chief factor of G. Using Theorems 1 and 4 we prove

Theorem 5

Let G be a finite group. Then $\mathrm{geom}(G, G) \leq \mathrm{ccf}(G) \cdot (\mathrm{ncf}(G))^{1/p}$ where p is the smallest prime factor of the order of $G/F(G)$.

A consequence of Theorem 5

Let us recall Theorem 5.

Theorem 5

Let G be a finite group. Then $\mathrm{geom}(G,G) \le \mathrm{ccf}(G) \cdot (\mathrm{ncf}(G))^{1/p}$ where p is the smallest prime factor of the order of $G/F(G)$.

By taking the reciprocals of both sides of the inequality of Theorem 5 and multiplying by $|G|$, we obtain the following result.

Corollary 6

Let G be a finite group. Then $\mathrm{ncf}(G) \le \left(\prod_{g\in G} |\mathrm{cl}_G(g)|\right)^{p/((p-1)|G|)}$ where p is the smallest prime factor of the order of $G/F(G)$.

BFC groups

A group is said to be a BFC group if its conjugacy classes are finite and of bounded size. A group G is called an n-BFC group if it is a BFC group and the least upper bound for the sizes of the conjugacy classes of G is n. One of B. H. Neumann's discoveries, dating back to 1954, was that in a BFC group the commutator subgroup G' is finite.

For all theorems in this talk about BFC groups, one can assume that the groups are finite. (This observation probably goes back to the 1950's.)

Note that $C_G(G')$ is a finite index nilpotent subgroup in a BFC group G. Thus, $F(G)$ is of finite index in the BFC group G.

A solution of another conjecture of Neumann and Vaughan-Lee

Let us recall Corollary 6 before we state our first result on BFC groups.

Corollary 6

Let G be a finite group. Then $\mathrm{ncf}(G) \leq \left(\prod_{g\in G} |\mathrm{cl}_G(g)|\right)^{p/((p-1)|G|)}$ where p is the smallest prime factor of the order of $G/F(G)$.

By noticing that $|\mathrm{cl}_G(1)| = 1$, Corollary 6 plus a bit more implies

Theorem 7

Let G be an n-BFC group with $n > 1$. Then $\mathrm{ncf}(G) < n^{p/(p-1)} \leq n^2$, where p is the smallest prime factor of the order of $G/F(G)$.

Theorem 7 solves a problem of Neumann and Vaughan-Lee from 1977.

History on BFC groups

Not long after B. H. Neumann's proof that the commutator subgroup G' of a BFC group is finite, Wiegold produced a bound for $|G'|$ for an n-BFC group G in terms of n and conjectured that $|G'| \leq n^{(1/2)(1+\log n)}$ where the logarithm is to base 2. Later Macdonald showed that $|G'| \leq n^{6n(\log n)^3}$ and Vaughan-Lee proved Wiegold's conjecture for nilpotent groups. For solvable groups the best bound to date is $|G'| \leq n^{(1/2)(5+\log n)}$ obtained by Neumann and Vaughan-Lee. In the same paper they showed that for an arbitrary n-BFC group G we have $|G'| \leq n^{(1/2)(3+5\log n)}$. Using the Classification of Finite Simple Groups Cartwright improved this bound to $|G'| \leq n^{(1/2)(41+\log n)}$ which was later further sharpened by Segal and Shalev who obtained $|G'| \leq n^{(1/2)(13+\log n)}$.

An improved bound for BFC groups

Let us recall Theorem 7.

Theorem 7

Let G be an n-BFC group with $n > 1$. Then $\mathrm{ncf}(G) < n^{p/(p-1)} \leq n^2$, where p is the smallest prime factor of the order of $G/F(G)$.

Applying Theorem 7 at the bottom of Page 511 of the Segal-Shalev paper we obtain the best general bound to date for the order of the commutator subgroup of a BFC group.

Theorem 8

Let G be an n-BFC group with $n > 1$. Then $|G'| < n^{(1/2)(7+\log n)}$.

Outer commutator words

A word ω is an element of a free group of finite rank. If the expression for ω involves k different indeterminates, then for every group G, we obtain a function from G^k to G by substituting group elements for the indeterminates. Thus we can consider the set G_ω of all values taken by this function. The subgroup generated by G_ω is called the verbal subgroup of ω in G and is denoted by $\omega(G)$.
An outer commutator word is a word obtained by nesting commutators but using always different indeterminates.
Fernández-Alcober and Morigi proved that if ω is an outer commutator word and G is any group with $|G_\omega| = m$ for some positive integer m then $|\omega(G)| \leq (m-1)^{m-1}$. They suspect that this bound can be improved to a bound close to one obtainable for the commutator word $\omega = [x_1, x_2]$. Theorem 8 yields

Theorem 9

Let G be a group with m commutators for some positive integer m at least 2. Then $|G'| < m^{(1/2)(7+\log m)}$.

BFC groups with no non-trivial abelian normal subgroup

Segal and Shalev showed that if G is an n-BFC group with no non-trivial abelian normal subgroup then $|G| < n^4$. We improve and generalize this result. For a finite group X let $k(X)$ denote the number of conjugacy classes of X.

Theorem 10

Let G be an n-BFC group with $n > 1$. If the Fitting subgroup $F(G)$ of G is finite, then $|G| < n^2 k(F(G))$. In particular, if G has no non-trivial abelian normal subgroup then $|G| < n^2$.

This essentially follows from a theorem of Guralnick and Robinson.

BFC groups with a given number of generators

The final main result concerns n-BFC groups with a given number of generators. Segal and Shalev proved that in such groups the order of the commutator subgroup is bounded by a polynomial function of n. In particular they obtained the bound $|G'| \leq n^{5d+4}$ for an arbitrary n-BFC group G that can be generated by d elements. By applying Theorem 7 to Page 515 of the Segal-Shalev paper we may improve this result.

Corollary 11

Let G be an n-BFC group that can be generated by d elements. Then $|G'| \leq n^{3d+2}$.

A quick consequence of Corollary 11

Finally, the following immediate consequence of Corollary 11 sharpens the last result in the Segal-Shalev paper.

Corollary 12

Let G be a d-generator group. Then

$$|\{[x,y] : x,y \in G\}| \geq |G'|^{1/(3d+2)}.$$

Thank you for your attention!