# Pro-$p$ Groups with Waists

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# Outline

# Waists are Comparable Subgrups

**Definition**

Let $G$ be a pro-$p$ group.
A subgroup $W$ of G is said to be a waist if for every open normal subgroup $N$ of $G$ we have $W \geq N$ or $N \geq W$.

The subgroups 1 and $G$ itself are trivial waists. From now on we shall not consider them.
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# First Properties of Waists

If $W$ is a waist then

- $W$ is a characteristic subgroup of $G$ (in particular $W$ is a normal subgroup)
- $W$ has finite index in $G$
- $G$ is finitely generated

When $G$ is a finite $p$-group only the first property is significant. In order to state the results in a simpler way, from now on we shall consider just finite $p$-groups (well, almost).

**Proposition**

*Let $W$ be a normal subgroup of $G$ of index $p^n$. Then $W$ is a waist if and only if it is the only normal subgroup of index $p^n$.*

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# Some Simple Examples

- All the subgroups of a (pro-)cyclic group are waists
- The finite $p$-groups with cyclic centre are those which have exactly one normal subgroup of order $p$. These groups are called monolithic and they have been extensively studied
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- In 1974 M. Auslander, E. L. Green and I. Reiten introduced the notion of waists in modules over Artin rings
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# Obliquity zero

- All the terms of the lower central series of a $p$-group of maximal class are waists



From left to right we have normal subgroups of increasing indices

- All the terms of the lower central series of the Nottingham group are waists



If all the terms of the lower central series are waists, the group is said to be of obliquity zero. The obliquity seems to be one of the most interesting invariants for the 'classification' of finite $p$-groups.

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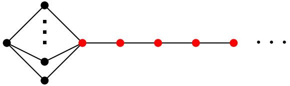

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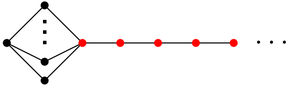

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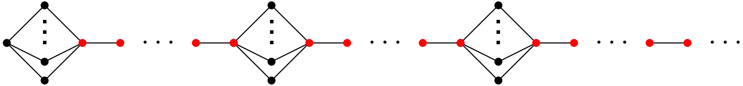

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  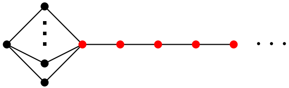

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  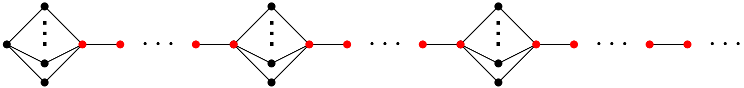

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# Camina Kernels are Waists

## Definition

A normal subgroup $K$ of a finite group $G$ is said to be a Camina kernel if, for every element $x$ not in $K$, the elements of the coset $xK$ are conjugate.

It is easy to show that Camina kernel are waists.

Camina kernels in $p$-groups have been extensively studied: A. Camina (1978), Macdonald (1981-1986), Chillag and Macdonald (1984), Chillag, Mann and Scoppola (1988), Dark and Scoppola (1996).

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# Waists Belong to the Upper and Lower Central Series

All the examples of waists we are aware of are terms of both the lower and upper central series of the group (with two obvious exceptions: the subgroups of a cyclic $p$-group and the unique normal subgroup of order $p$ in a monolithic $p$-group).

**Theorem**

*Let $G$ be a non-cyclic finite $p$-group, with $p \neq 2$. If $W$ is a waist of order not $p$ then $W$ is a term of the lower central series of $G$.*

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Despite the similarity of the statements the proofs of these theorems are quite different.

# The First (Common) Step of the Proofs

- If $W$ is a waist which is not a term of the lower central series of $G$ then $W$ lies between two consecutive terms of the lower central series:

$$G_i \gneq W \gneq G_{i+1} \text{ for some integer } i.$$

- If $W$ is a waist which is not a term of the upper central series of $G$ then $W$ lies between two consecutive terms of the upper central series:

$$Z_{j+1}(G) \gneq W \gneq Z_j(G) \text{ for some integer } j.$$

- Consider two normal subgroups $H$ and $L$ such that $H \gneq W \gneq L$ and $(H, G) \leq L$.
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- We have a waist $W$ and two normal subgroups $H$ and $L$ such that $H \gneq W \gneq L$ and $(H, G) \leq L$.
- Under this hypothesis we get a number of structural results: in particular $H/L$ is a cyclic group of order $p^2$.
- This means that $W$ is 'nearly' a term of every central series.
- This situation actually occurs: it is not true that $W$ is a term of every central series.
- The two theorems about the upper and the lower central series are then really different.

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# When the Derived Subgroup is a Waist

If the derived subgroup $G_2$ of a finite $p$-group $G$ is a Camina kernel then $G$ has nilpotency class at most 3 and $G_3$ is a Camina kernel too. (Macdonald 1981, Dark and Scoppola 1994)

**Theorem**

*Let $p \neq 2$. If the derived subgroup $G_2$ is a waist then the third term $G_3$ of the lower central series is a waist too.*

If the derived subgroup is a waist, the nilpotency class of $G$ needs not to be limited.

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# Open Problems

## The Case $p = 2$

All our main result need the hypothesis $p \neq 2$. Study the case $p = 2$.

## Waists from Waists

- We proved that if $G_2$ is a waist then $G_3$ is a waist.
- We know that if $G$ is a 2-generated metabelian $p$-group and $G_2$, $G_3$ and $G_4$ are waists then every term of the lower central series of $G$ is a waist. (Brandl, Caranti and Scoppola - 1992)
- Finding more theorems like these: if you have enough waists (possibly in prescribed positions) then there are other waists.

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- Finding more theorems like these: if you have enough waists (possibly in prescribed positions) then there are other waists.

# Open Problems

## The Case $p = 2$

All our main result need the hypothesis $p \neq 2$. Study the case $p = 2$.

## Waists from Waists

- We proved that if $G_2$ is a waist then $G_3$ is a waist.
- We know that if $G$ is a 2-generated metabelian $p$-group and $G_2$, $G_3$ and $G_4$ are waists then every term of the lower central series of $G$ is a waist. (Brandl, Caranti and Scoppola - 1992)
- Finding more theorems like these: if you have enough waists (possibly in prescribed positions) then there are other waists.