Unramified Brauer groups of finite and infinite groups

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Stable rationality

A field extension K/k is **stably rational** if there exist r and s such that

$$K(x_1, \dots, x_r) \cong k(y_1, \dots, y_s).$$

Let k be an algebraically closed field of characteristic 0, V a finite dimensional vector space over k. Let $G \subset \mathrm{GL}(V)$ be a finite group (or a reductive group acting almost freely on V).

Question

When is the field of invariants $k(V)^G$ (stably) rational over k?

By the **no-name lemma**, the answer does not depend on V, but only on G itself.

$k(V)^G/k$ stably rational – examples

Positive answer

- Abelian groups, S_n, A_5.
- All groups of order p^n, $n \leq 4$.
- Special groups: $\mathrm{GL}_n(k)$, $\mathrm{SL}_n(k)$, $\mathrm{Sp}_n(k)$.
- Orthogonal groups: $\mathrm{O}_n(k)$, $\mathrm{SO}_n(k)$.
- $\mathrm{PGL}_n(k)$ if n divides 420.

Counterexamples

- Saltman (1984). Counterexamples of G of order p^9.
- Bogomolov (1988). Counterexamples of G of order p^6.

Some open cases

- G finite nonabelian simple.
- G connected.

The unramified Brauer group – finite case

Artin, Mumford (1972) introduced the **unramified Brauer group** $\mathrm{H}^2_{\mathrm{nr}}(k(V)^G, \mathbb{Q}/\mathbb{Z})$. It is a subgroup of $\mathrm{Br}\, k(V)^G$. If

$$\mathrm{H}^2_{\mathrm{nr}}(k(V)^G, \mathbb{Q}/\mathbb{Z}) \neq 0,$$

then $k(V)^G/k$ is **not** stably rational.

Bogomolov (1988). if G is finite, then $\mathrm{H}^2_{\mathrm{nr}}(k(V)^G, \mathbb{Q}/\mathbb{Z}) \cong \mathrm{B}_0(G)$, where

$$\mathrm{B}_0(G) = \bigcap_{\substack{A \leq G,\\ A \text{ abelian}}} \ker \mathrm{res}^G_A,$$

where $\mathrm{res}^G_A : \mathrm{H}^2(G, \mathbb{Q}/\mathbb{Z}) \to \mathrm{H}^2(A, \mathbb{Q}/\mathbb{Z})$ is the usual cohomological restriction map.

The unramified Brauer group – reductive case

If G is reductive, the definition of $\mathrm{B}_0(G)$ needs to be modified:

Define

$$K_G = \{\gamma \in \mathrm{H}^2(G, \mathbb{Q}/\mathbb{Z}) \mid \mathrm{res}^G_H \gamma = 0 \text{ for every finite } H \leq G\}.$$

Let

$$\mathrm{B}_0(G) = \{\gamma + K_G \in \mathrm{H}^2(G, \mathbb{Q}/\mathbb{Z})/K_G \mid \mathrm{res}^G_A \gamma = 0 \text{ for every finite abelian } A \leq G\}.$$

Bogomolov (1988). $\mathrm{B}_0(G) \cong \mathrm{H}^2_{\mathrm{nr}}(k(V)^G, \mathbb{Q}/\mathbb{Z})$, where V is any generically free representation of G.

Computing $\mathrm{B}_0(G)$

Saltman (1984). $\mathrm{B}_0(\mathrm{PGL}_n(k)) = 0$.

Bogomolov, Maciel, Petrov (2004). If G is a finite simple group of Lie type, then $\mathrm{B}_0(G) = 0$.

Chu, Hu, Kang, Prokhorov (2009). $\mathrm{B}_0(G) = 0$ for all groups G of order 32.

Chu, Hu, Kang, Kunyavskiĭ (2009). $\mathrm{B}_0(G)$ for all groups G of order 64. Nine of these have nontrivial B_0.

Kunyavskiĭ (2010). If G is a finite simple group, then $\mathrm{B}_0(G) = 0$.

The nonabelian exterior square of a group

Let G be a group. We form a group $G \wedge G$, generated by the symbols $m \wedge n$, where $m, n \in G$, subject to the following relations:

$$
\begin{aligned}
mm' \wedge n &= ({}^m m' \wedge {}^m n)(m \wedge n), \\
m \wedge nn' &= (m \wedge n)({}^n m \wedge {}^n n'), \\
m \wedge m &= 1,
\end{aligned}
$$

for all $m, m', n, n' \in G$.

Miller (1952). $\mathrm{M}(G) = \ker(G \wedge G \to [G, G])$ is naturally isomorphic to $\mathrm{H}_2(G, \mathbb{Z})$.

Homological description of $\mathrm{B}_0(G)$ – finite case

Identify $\mathrm{H}_2(G,\mathbb{Z})$ with $\mathrm{M}(G)$. Set

$$\mathrm{M}_0(G) = \langle \mathrm{cor}_G^A \mathrm{M}(A) \mid A \leq G, A \text{ abelian}\rangle.$$

It turns out that $\mathrm{M}_0(G) = \langle x \wedge y \mid x, y \in G, [x, y] = 1\rangle$.

Theorem

Let G be a finite group. Then $\mathrm{B}_0(G)$ is naturally isomorphic to

$$\mathrm{Hom}(\mathrm{M}(G)/\,\mathrm{M}_0(G), \mathbb{Q}/\mathbb{Z}),$$

hence $\mathrm{B}_0(G) \cong \mathrm{M}(G)/\,\mathrm{M}_0(G)$ (non-canonically).

Homological description of $\mathrm{B}_0(G)$ – reductive case

If G is reductive, set

$$\bar{\mathrm{M}}(G) = \langle \mathrm{cor}_G^H \mathrm{M}(H) \mid H \leq G, |H| < \infty \rangle$$

and

$$\begin{aligned} \bar{\mathrm{M}}_0(G) &= \langle \mathrm{cor}_G^A \mathrm{M}(A) \mid A \leq G, |A| < \infty, A \text{ abelian} \rangle \\ &= \langle x \wedge y \mid [x, y] = 1, |x| < \infty, |y| < \infty \rangle. \end{aligned}$$

Theorem

If G is a reductive group, then $\mathrm{B}_0(G)$ is naturally isomorphic to

$$\mathrm{Hom}(\bar{\mathrm{M}}(G)/\bar{\mathrm{M}}_0(G), \mathbb{Q}/\mathbb{Z}).$$

Let G be any group. From here on we write

$$\mathrm{B}_0(G) = \frac{\mathrm{M}(G)}{\mathrm{M}_0(G)} \quad \text{and} \quad \bar{\mathrm{B}}_0(G) = \frac{\bar{\mathrm{M}}(G)}{\bar{\mathrm{M}}_0(G)}.$$

B_0 vs $\bar{\mathrm{B}}_0$

Theorem

Let G be a locally finite group. Then $\mathrm{B}_0(G) \cong \bar{\mathrm{B}}_0(G)$.

Example

Suppose $m > 1$ and let $n > 2^{48}$ be odd. Let F be a free group of rank m. Let $G = F/F^n$ be the **free Burnside group** of rank m and exponent n. Then $\bar{\mathrm{B}}_0(G) = 0$ and $\mathrm{B}_0(G) \cong \mathrm{H}_2(G, \mathbb{Z})$ is free abelian of countable rank.

Example

If G is a **one-relator group with torsion**, then $\bar{\mathrm{B}}_0(G) = 0$ by Newman's description of finite subgroups of G. Since all centralizers of nontrivial elements of G are cyclic, $\mathrm{M}_0(G) = 0$ and therefore $\mathrm{B}_0(G) \cong \mathrm{H}_2(G, \mathbb{Z})$. The latter can be nontrivial (Lyndon, 1950).

Hopf formula and 5-term B_0-sequence

For a group G let $K(G)$ be the **set** of all commutators in G.

Theorem

Let G be a group given by a free presentation $G = F/R$. Then

$$B_0(G) \cong \frac{\gamma_2(F) \cap R}{\langle K(F) \cap R \rangle}.$$

Theorem

Let G be a group and N a normal subgroup of G. Then we have the following exact sequence:

$$B_0(G) \to B_0(G/N) \to \frac{N}{\langle K(G) \cap N \rangle} \to G^{\mathrm{ab}} \to (G/N)^{\mathrm{ab}} \to 0.$$

Some consequences

Explicit descriptions of $\mathrm{B}_0(G)$ can be obtained for some G:

- G is a p-group of class 2,
- G is a split extension (in particular, Frobenius group),

$\mathrm{B}_0(G)$ is related to special types of central extensions:

A central extension (E, π, A) of a group G is a **CP-extension** if commuting elements of G lift to commuting elements in E.
A CP-extension (U, ϕ, A) of G is **CP-universal** if for every CP-extension (E, ψ, B) of G there exists a homomorphism $\chi : U \to E$ that factors through G.

Theorem

A group G admits a CP-universal central extension if and only if it is perfect. In the latter case, $((G \wedge G)/\,\mathrm{M}_0(G)/, \kappa, \mathrm{B}_0(G))$ is the CP-universal central extension of G.

Computing $B_0(G)$ when G is polycyclic

Eick, Nickel (2008). Algorithm for computing $G \wedge G$ when G is polycyclic. This allows efficient computations of

$$M(G) = \ker(G \wedge G \to [G, G])$$

and

$$M_0(G) = \langle x \wedge y \mid x, y \in G, [x, y] = 1 \rangle,$$

and hence $B_0(G) = M(G)/M_0(G)$.
Can compute $B_0(G)$ for moderately large finite solvable groups G, and some infinite polycyclic groups. For groups of small order the results coincide with hand calculations. **But**:

Bogomolov (1988) claimed that if $|G| = p^5$, then $B_0(G) = 0$.

We have found three groups of order 243 with $B_0(G) \neq 0$. For these groups $k(V)^G/k$ is not stably rational.

Computational data

All solvable groups G of order ≤ 729, apart from the orders 512, 576 and 640, with $\mathrm{B}_0(G) \neq 0$.

n	# of groups of order n	# of G with $\mathrm{B}_0(G) \neq 0$
64	267	9
128	2328	230
192	1543	54
243	67	3
256	56092	5953
320	1640	54
384	20169	1820
448	1396	54
486	261	3
704	1387	54
729	504	85

Table: Numbers of groups G with $\mathrm{B}_0(G) \neq 0$.

B_0 in K-theory

Let Λ be a ring with 1. Let $\mathrm{E}(\Lambda) \leq \mathrm{GL}(\Lambda)$ be generated by all **elementary matrices**, and let $\mathrm{St}(\Lambda)$ be the **Steinberg group**. The K_2 **functor** is defined by $\mathrm{K}_2\,\Lambda = Z(\mathrm{St}(\Lambda))$. It is known that $\mathrm{K}_2\,\Lambda \cong \mathrm{H}_2(\mathrm{E}(\Lambda), \mathbb{Z})$.
Let $A, B \in \mathrm{E}(\Lambda)$ commute, and choose their preimages $a, b \in \mathrm{St}(\Lambda)$. Define $A \star B = [a, b] \in \mathrm{K}_2\,\Lambda$ to be the **Milnor element** induced by A and B.

Theorem

Denote $\mathrm{B}_0\,\Lambda = B_0(\mathrm{E}(\Lambda))$.

1. $\mathrm{B}_0\,\Lambda \cong \mathrm{K}_2\,\Lambda / \langle \mathrm{K}(\mathrm{St}(\Lambda)) \cap \mathrm{K}_2\,\Lambda \rangle$.
2. $\mathrm{B}_0\,\Lambda = 0$ *iff* $\mathrm{K}_2\,\Lambda$ *is generated by Milnor's elements.*
3. $\mathrm{B}_0\,\Lambda$ *is naturally isomorphic to* $\mathrm{B}_0(\mathrm{GL}(\Lambda))$.

Conjecture (equivalent to the Bass conjecture)

$\mathrm{B}_0\,\Lambda = 0$ *for every unital ring* Λ.