Advances on Brauer's Height Zero Conjecture

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I. Introduction.

II. Brauer's Height Zero Conjecture.

This is joint work with P. H. Tiep, from the University of Arizona

This talk is dedicated to the memory of Silvia Lucido

Let G be a finite group and let p be a prime.

The local subgroups of G are

$$\mathbf{N}_G(Q)$$

for p-subgroups $1 < Q$ of G.

MAIN IDEA

Find the relation between the local and global structure of G.

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A perfect example:

Theorem (Frobenius)

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THEOREM (FROBENIUS)

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Much of the representation theory of finite groups is devoted to proving some extraordinary global/local conjectures

1. Counting Conjectures.
2. Brauer's Conjectures.

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CONJECTURE (McKay, 1971)

If G is a finite group and $P \in \mathrm{Syl}_p(G)$, then

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If G is a finite group, then

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We also have Dade's Conjectures (1992-1994) which locally count $\chi \in \mathrm{Irr}(G)$ with a fixed $\chi(1)_p = p^d$.

2. Brauer's Conjectures

There are many ways to introduce Richard Brauer blocks. One of the character theoretic approaches is: link

$$\alpha \leftrightarrow \beta$$

if and only if

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The blocks are the connected components of this linking.

If B is a block, then $\mathrm{Irr}(B)$ denotes the irreducible characters in the block B. The connected component of the principal character 1_G is called the **principal block**. In some important cases, G only has the principal block.

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Let G be a finite group of order dividing n, R are the integers in $\mathbb{Q}_n$, $\mathcal{P}$ is a maximal ideal of R containing p, and $F = R/\mathcal{P}$, a field of characteristic p. Then

$$FG = B_1 \oplus \cdots \oplus B_s$$

where the B_i's are indecomposable ideals.

Write $^* : R \to F$ for the canonical homomorphism, and

$$1 = e_1 + \cdots + e_s \,,$$

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The connection between characteristic zero and characteristic p is what makes the subject deep. If $\chi \in \mathrm{Irr}(G)$, let

$$\omega_\chi : \mathbf{Z}(RG) \to R$$

be the algebra homomorphism

$$\omega_\chi(\hat{K}) = \frac{\chi(x_K)|K|}{\chi(1)}$$

where $K = \mathrm{cl}_G(x_K)$ is any conjugacy class of G, $\hat{K} = \sum_{x \in K} x$. Then χ belongs to the block B_i if and only if $(\omega_\chi)^* = \lambda_i$, the unique such that $\lambda_i(e_j) = \delta_{ij}$.

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So we also have that $\alpha, \beta \in \mathrm{Irr}(G)$ belong to the same block if and only if

$$\left(\frac{\alpha(x_K)|K|}{\alpha(1)}\right)^* = \left(\frac{\beta(x_K)|K|}{\beta(1)}\right)^*$$

for every conjugacy class K of G.

All Brauer's Conjectures deal with defect groups.

Every block B of G has canonically associated a conjugacy class of p-subgroups D of G. We say that D is a **defect group** of B (or that $B \in \mathrm{Bl}(G|D)$). We write $|D| = p^d$ and call d the **defect** of B.

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For instance, the defect groups D of the principal block are the Sylow p-subgroups of G. (When $D \in \mathrm{Syl}_p(G)$, then we say that B has **maximal defect**.) In other extreme case, it can be proved that B has defect group $D = 1$ (i.e. $d = 0$) if and only if $\mathrm{Irr}(B) = \{\chi\}$ which happens if and only if $\chi(1)_p = |G|_p$.

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Another perfect example of the global/local theory:

Brauer's First Main Theorem

If D is a p-subgroup of G, then there exists a canonical bijection

$$b \mapsto b^G$$

from

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The influence of the defect group D.

- If $\chi \in \mathrm{Irr}(B)$, then $\chi(g) = 0$ if $g_p \notin_G D$.
- $\min_{\chi \in \mathrm{Irr}(B)}\{\chi(1)_p\} = |G|_p/|D|$. (So there is $\chi \in \mathrm{Irr}(B)$ of p'-degree if and only if $D \in \mathrm{Syl}_p(G)$.)
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Brauer's $k(B)$-conjecture

If B is a block with defect group D, then

$$k(B) = |\mathrm{Irr}(B)| \leq |D| .$$

If G is p-constrained (i. e., $\mathbf{C}_G(\mathbf{O}_p(G)) \subseteq \mathbf{O}_p(G)$), then G has a unique block (the principal block). Hence

Consequence

If G is p-constrained and $P \in \mathrm{Syl}_p(G)$, then

$$k(G) \leq |P| .$$

Blocks are the right thing to look at: It is false that $k(G) \leq |P|$, in general.

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The $k(B)$-conjecture for p-solvable groups has remained a challenge for many years. It easily reduces (Nagao) to proving:

The $k(GV)$-Theorem (2004)

If V is a finite faithful G-module of characteristic p, and G is a p'-group, then

$$k(GV) \leq |V| .$$

This is an incredibly difficult theorem, only possible because of the work of R. Knörr, then G. Robinson and J. Thompson, and finally D. Gluck, K. Magaard, U. Riese and P. Schmid.

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Suppose that B is a block with defect group D. Write $|G|_p = p^a$ and $|D| = p^d$. We already know that

$$\min_{\chi \in \mathrm{Irr}(B)} \{\chi(1)_p\} = p^{a-d}\,.$$

So if $\chi \in \mathrm{Irr}(B)$, then

$$\chi(1)_p = p^{a-d+h}$$

where $h \geq 0$ is called the **height** of χ.

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Brauer's Height Zero Conjecture

Let B be a block with defect group D. Then all $\chi \in \mathrm{Irr}(B)$ have height zero if and only if D is abelian.

Both directions are quite deep (*specially* the "only if" direction; i. e., proving that D is abelian). In the important case where $D \in \mathrm{Syl}_p(G)$, then it says that all $\chi \in \mathrm{Irr}(B)$ have p'-degree if and only if D is abelian.

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Again blocks are the right thing to look at: if p does not divide $\chi(1)$ for ALL $\chi \in \mathrm{Irr}(G)$, then $P \in \mathrm{Syl}_p(G)$ is abelian AND normal in G.

Brauer's Height Zero has a wonderful consequence.

If B is the principal block of G, it is easy to check that $\chi \in \mathrm{Irr}(B)$ if and only if $\sum_{x \in G^0} \chi(x) \neq 0$.

CONSEQUENCE

Let $P \in \mathrm{Syl}_p(G)$. Then P is abelian if and only if whenever $\chi \in \mathrm{Irr}(G)$ has degree divisible by p, then $\sum_{x \in G^0} \chi(x) = 0$.

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For p-solvable groups, the "if" direction is almost trivial. The "only if" direction is a very deep theorem which easily reduces to proving:

The Gluck-Wolf Theorem (1984)

Let G be a p-solvable group, $N \triangleleft G$, $\theta \in \mathrm{Irr}(N)$ be G-invariant. Let $P/N \in \mathrm{Syl}_p(G/N)$. If $\chi(1)/\theta(1)$ is not divisible by p for all $\chi \in \mathrm{Irr}(G|\theta)$, then P/N is abelian.

Previous results.

- In 1984, D. Gluck and T. Wolf proved the conjecture for p-solvable groups.
- In 1988, T. Berger and R. Knörr reduced the "if direction" to quasisimple groups.
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This is our main result here.

THEOREM (N-TIEP)

Let $p = 2$ and let B be a p-block with maximal defect. Then Brauer's Height Zero Conjecture is true for B.

COROLLARY

Let $P \in \mathrm{Syl}_2(G)$. Then P is abelian if and only if whenever $\chi \in \mathrm{Irr}(G)$ has even degree then $\sum_{x \in G^0} \chi(x) = 0$.

Until the end of the talk, I will try to explain:

- Why we have been able to prove this result now.
- Why $p = 2$, why maximal defect.
- How the proof goes.

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There are three obstacles to the theorem.

In an inductive proof (on $|G : Z|$, where $Z = \mathbf{O}_{p'}(G)$), we can easily assume that $Z \subseteq \mathbf{Z}(G)$ has p'-order, and that proper normal subgroups of G have abelian Sylow p-subgroups.

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Now, let N/Z be a minimal normal subgroup of G/Z.

If N/Z is abelian, then induction and delicate arguments show that $\mathrm{Irr}(B) = \mathrm{Irr}(G|\lambda)$, for some $\lambda \in \mathrm{Irr}(Z)$. So we have that p does not divide $\chi(1)$ for all $\chi \in \mathrm{Irr}(G|\lambda)$ and we wish to show that a Sylow p-subgroup of G is abelian.

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OBSTACLE B. Prove the Gluck-Wolf Theorem for finite groups:

CONJECTURE

Let G be a finite group, $N \lhd G$, $\theta \in \mathrm{Irr}(N)$ be G-invariant. Let $P/N \in \mathrm{Syl}_p(G/N)$. If $\chi(1)/\theta(1)$ is not divisible by p for all $\chi \in \mathrm{Irr}(G|\theta)$, then P/N is abelian.

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There is no way to avoid this: It is implied by Brauer's Height Zero Conjecture.

This is one of the great challenges in character theory. As in the p-solvable case, it leads to study and classify finite G-modules V of characteristic p where all the orbits of $v \in V$ have p'-size.

A big team (including M. Liebeck, N, J. Saxl, P. Tiep, and collaborators) are about to classify these actions. Once this is done, we will be able to prove the Gluck-Wolf theorem for all finite groups.

How did we overcome this difficulty for $p = 2$? We were fortunate to have

Theorem (A. Moretó)

Let G be a finite group, $N \lhd G$, $\theta \in \mathrm{Irr}(N)$ be G-invariant. If $\chi(1)/\theta(1)$ is odd for all $\chi \in \mathrm{Irr}(G|\theta)$, then G/N is solvable. In particular, G/N has abelian Sylow 2-subgroups.

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```
Display(CharacterTable("2.A5"));
2.A5

     2  3  3  2  1  1   1   1   1   1
     3  1  1  .  1  1   .   .   .   .
     5  1  1  .  .  .   1   1   1   1

       1a 2a 4a 3a 6a  5a 10a  5b 10b
    2P 1a 1a 2a 3a 3a  5b  5b  5a  5a
    3P 1a 2a 4a 1a 2a  5b 10b  5a 10a
    5P 1a 2a 4a 3a 6a  1a  2a  1a  2a

X.1     1  1  1  1  1   1   1   1   1
X.2     3  3 -1  .  .   A   A  *A  *A
X.3     3  3 -1  .  .  *A  *A   A   A
X.4     4  4  .  1  1  -1  -1  -1  -1
X.5     5  5  1 -1 -1   .   .   .   .
X.6     2 -2  . -1  1  -A   A -*A  *A
X.7     2 -2  . -1  1 -*A  *A  -A   A
X.8     4 -4  .  1 -1  -1   1  -1   1
X.9     6 -6  .  .  .   1  -1   1  -1

A = -E(5)-E(5)^4
  = (1-ER(5))/2 = -b5
```

In the final case, N/Z is non-abelian; hence it is a direct product of simple groups with abelian Sylow p-subgroups.

For $p = 2$, again we were fortunate to have a recent theorem that says that simple groups with abelian Sylow 2-subgroups are *good* for the McKay conjecture (in the Isaacs-Malle-N, IMN) sense. (This theorem proved the McKay conjecture for groups with abelian Sylow 2-subgroups.)

This means that we have bijections

$$^* : \mathrm{Irr}_{2'}(N|\lambda) \to \mathrm{Irr}_{2'}(\mathbf{N}_N(Q)|\lambda)\,,$$

where $Q \in \mathrm{Syl}_2(N)$, such the character theory of G over θ looks like the character theory of $\mathbf{N}_G(Q)$ over θ^*.

With this bijection ($p = 2$) in hand and quite delicate arguments we could prove that the only block b of $\mathbf{N}_G(Q)$ that induces B satisfies the hypothesis. Then induction.

Our work (IMN) was focused on the the McKay conjecture, not in its block version, the so called **Alperin-McKay conjecture**:

Alperin-McKay Conjecture

If B and b are Brauer First Main correspondents, then B and b have equal number of height zero characters.

If now we are dealing with blocks, we should incorporate them to the IMN reduction (at least for blocks of maximal defect).

OBSTACLE C. Define what is a good simple group for the Alperin-McKay conjecture (at least for blocks of maximal defect). Prove that simple groups with abelian Sylow p-subgroups are good for Alperin-Mckay.

Obstacle C is not asking little. It says that we have to prove (a strong form) of the Alperin-McKay conjecture for groups with abelian Sylow p-subgroups before we can prove Brauer's Height Zero Conjecture for blocks of maximal defect.

We had that result for $p = 2$ thanks to the recent work in IMN. For p odd, this is the last obstacle to achieve Brauer's Height Zero Conjecture for blocks of maximal defect.