Localizations of finitely generated soluble groups

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Localization

Localization: A functor which assigns to each group G a unique π-local group G_π (where π is a set of primes).

Definition

A group G is said to be π-local if the map

$$x \mapsto x^q$$

is bijective for every prime $q \notin \pi$.

A π-localization of any group G is a homomorphism

$$\phi_\pi : G \to G_\pi$$

where G_π is a π-local group, with the universal property that given any homomorphism $\alpha : G \to H$, with H $\pi-$local, there exists a unique homomorphism $\alpha_\pi : G_\pi \to H$ such that $\alpha_\pi \phi_\pi = \alpha$.

Localization

If $\alpha : G \to H$, where H is π-local, then we have the following commutative diagram:

$$\begin{array}{ccc} G & & \\ \phi_\pi \big\downarrow & \searrow \alpha & \\ G_\pi & \xrightarrow{\alpha_\pi} & H \end{array}$$

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$$\begin{array}{ccc} G & \xrightarrow{\alpha} & H \\ \phi_\pi \big\downarrow & & \big\downarrow \phi_\pi \\ G_\pi & \xrightarrow{\alpha_\pi} & H_\pi \end{array}$$

If A is an abelian normal subgroup of G, then the conjugation action of G on A induces an action of G on A_π.

Examples

Let π be a set of primes, we say $b \in \pi'$ if all the primes divisors of b are not in π. We will assume from now on that π is a proper subset of the set of all primes.

If G is a finite group, then

$$G_\pi = G/T_{\pi'}$$

where $T_{\pi'}$ is the subgroup generated by the π'-torsion elements of G.

$$\mathbb{Z}_\pi = \left\{ \frac{a}{b} \,\middle|\, a,\, b \in \mathbb{Z},\, b \in \pi' \right\}.$$

If $N \leq Z(G)$, then $(G/N)_\pi \cong G_\pi/N_\pi$

Nilpotent and virtually nilpotent groups

Theorem

If G is nilpotent of class c, then G_π is nilpotent of class $\leq c$ (Ribenboim 79).

If G is nilpotent, then $\mathrm{Ker}(G \to G_\pi) = T_{\pi'}$.

If G is nilpotent and $x \in G_\pi$, then there exists $m \in \pi'$ such that $x^m \in \phi_\pi(G)$.

If G is nilpotent and $N \triangleleft G$, then N_π embeds in G_π and it's image is normal. Furthermore $(G/N)_\pi = G_\pi / N_\pi$.

Theorem

If G is virtually nilpotent, then G_π is virtually nilpotent (Casacuberta and Castellet 92).

Morphisms

If $\alpha : G \to H$ is an epimorphism, then $\alpha_\pi : G_\pi \to H_\pi$ is also an epimorphism.

However localization does not usually preserve monomorphisms:

If $\pi = \{3\}$, then we have the embedding $i : C_3 \to S_3$,

$$(C_3)_\pi = C_3, \;\; (S_3)_\pi = 1.$$

So that i_π is not a monomorphism.

Soluble groups

What can we say about localization and soluble groups?

B. Neumann (59) constructed a group G with unique π'-roots which did not embed in G_π.

Baumslag (60) defined a class of groups that could be embedded in a π-local group. Free groups are in this class.

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Class $\mathcal{P}_\pi$

Definition

Let π be a set of primes. A group G is said to be in the class $\mathcal{P}_\pi$ if

(i) G has unique π'-roots;

(ii) if $x \in G$ has no m-th root, for some $m \in \pi'$, and

$$D_G(x) = \{g \in G \mid C_G(x) \cap C_G(x)^g \neq 1\},$$

then $C_G(x)$ is torsion-free, $D_G(x)$ is a group and one of the following holds:

(a) $D_G(x) = C_G(x) \lneq G$ so that $C_G(x)$ is malnormal;

(b) $|D_G(x) : C_G(x)| = 2$ and $C_G(x) \leq \mathbb{Z}_\pi$.

Theorem

If $G \in \mathcal{P}_\pi$, then G embeds in G_π. [Baumslag 60, Cassidy 88]

Soluble groups

Theorem

Let $G \in \mathcal{P}_\pi$ be a soluble group which is not π-local. Then G_π is not soluble.

Take $x \in G$ that has no m-th root, for some $m \in \pi'$, attach roots for x:

$$K = G *_{D_G(x)} P.$$

Then $G_\pi \cong K_\pi$, K embeds in G_π.
If $g \in K^{(t)}$, then there exists $h \in K$ such that $[g, g^h] \neq 1$.
K is not soluble and so G_π is not soluble.

Free Soluble groups

Theorem

If G is a free soluble group of derived length at least 2, then G_π is not soluble.

Let G be a free metabelian group and let $A = G'$. Then $C_G(x)$ is manormal, for all $x \notin A$. Set

$$H = (A_\pi \rtimes G)/T \text{ where } T = < (\phi_\pi(a), a^{-1}) \mid a \in A >^{Ncl} .$$

We have the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & A & \longrightarrow & G & \longrightarrow & Q & \longrightarrow & 1 \\ & & \phi_\pi \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & A_\pi & \longrightarrow & H & \longrightarrow & Q & \longrightarrow & 1. \end{array}$$

$G_\pi \cong H_\pi, \quad H \in \mathcal{P}_\pi \implies G_\pi$ is not soluble. As localization preserves epimorphisms, G_π is not soluble whenever G is a free soluble group of derived length at least 2.

Just Non-X groups

Definition

A group G is said to be just-non virtually nilpotent (JNVN) if G is not virtually nilpotent but every proper quotient of G is virtually nilpotent. [Robinson-Wilson 84, Zhang 91, De Falco 02]

Definition

A group G is said to be a $JNVN_\pi$ group if G has unique π' roots and is not virtually nilpotent but every proper quotient of G with unique π' roots is virtually nilpotent.

Metabelian groups

Theorem

Let G be a finitely generated metabelian $JNVN_\pi$ group with unique π'-roots. Then G_π is not soluble.

- If G is a finitely generated metabelian $JNVN_\pi$ group, then choose A maximal subject to being abelian and containing G'.
- If $A \lneq N \lhd G$, then $Z(N) = 1$.
- So $C_A(x) = 1$ and $C_G(x)$ is malnormal, for all $x \notin A$.
- A is either an elementary abelian p-group or is torsion-free.
- p-case: $G \in \mathcal{P}_\pi$.
- Torsion-free case: Construct H as in the free case.
- As before $G_\pi \cong H_\pi$, $H \in \mathcal{P}_\pi$.
- Hence G_π is not soluble.

Metabelian groups

Corollary

Let G be a finitely generated metabelian group. Then G_π is either virtually nilpotent or is not soluble.

Let G be a finitely generated metabelian group, then

either every quotient of G with unique π'-roots is virtually nilpotent and hence G_π is virtually nilpotent

or G has a $JNVN_\pi$ quotient and hence G_π is not soluble.

Soluble groups

Theorem

Let G be a finitely generated soluble group and let π be a set of primes such that π' is not finite. Then either G_π is virtually nilpotent or there exists a set of primes τ containing π such that $\tau \backslash \pi$ is a finite set and G_τ is not soluble.

Let G be a finitely generated soluble group. Then

either G is virtually nilpotent and so G_π is virtually nilpotent,

or G has a JNVN quotient.

In the second case $H = G^n$ is a finitely generated metabelian group, for some n.

Let $\tau = \pi \cup \pi(n)$.

Then G_τ is either virtually nilpotent and hence G_π is virtually nilpotent or G_τ is not soluble.