EXTERIOR SELF-QUOTIENT MODULES

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Unique Characteristic Subgroup

Definition. G is a **UCS-group** if it contains a unique non-trivial proper characteristic subgroup.

D. R. Taunt, Finite groups having unique proper characteristic subgroups I. *Proc. Cambridge Philos. Soc.* **51** (1955), 25–36.

Our setting: P a finite UCS p-group

Characteristically simple p-group $=$ elementary abelian p-group

UCS p-group: The only characteristic subgroups are $1 < \Phi(P) < P$.

$\Phi(P)$ is elementary abelian

Three cases

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We deal with nonabelian UCS p-groups

Then $\Phi(P) = \mathbf{Z}(P) = P'$

$x^p \in \Phi(P)$, so $x^{p^2} = 1$

$(xy)^p = x^p y^p [y, x]^{\binom{p}{2}} = x^p y^p$ if $p > 2$

- Case 1: $p = 2$
- Case 2: $p > 2$, P has exponent p
- Case 3: $p > 2$, P has exponent p^2

Action of the automorphism group

Let $V = P/\Phi(P)$ and $W = \Phi(P)$.

Both are vector spaces over the p-element field.

By the UCS property $\mathrm{Aut}(P)$ acts irreducibly both on V and W.

The kernel of the action on V is $\mathbf{O}_p(\mathrm{Aut}(P))$.

Let the image of the action be $G \leq \mathrm{GL}(V)$.

The action of G on $V = P/\mathbf{Z}(P)$ determines its action on $W = P'$.

We classify nonabelian UCS p-groups where the number of generators is at most 4, i.e., if $\dim P/\Phi(P) \leq 4$.

UCS-groups with $p = 2$

$\dim P/\Phi(P)$	$\dim \Phi(P)$	$\mathrm{Aut}(P)$ on $P/\Phi(P)$
2	1	$\mathrm{GL}_2(2)$
3	3	$\Gamma\mathrm{L}_1(2^3)$
4	1	$\mathrm{O}_4^+(2)$
4	1	$\mathrm{O}_4^-(2)$
4	2	$\mathrm{GL}_2(2) \otimes \mathrm{GL}_2(2)$
4	2	$\Gamma\mathrm{L}_1(2^4)$
4	4	$\mathrm{O}_4^+(2)$
4	4	$5 \cdot 4$
4	4	5
4	4	$\Gamma\mathrm{L}_2(2^2)$

UCS-groups with exponent p, $p > 2$

$\dim P/\Phi(P)$	$\dim \Phi(P)$	$\mathrm{Aut}(P)$ on $P/\Phi(P)$
2	1	$\mathrm{GL}_2(p)$
3	3	$\mathrm{GL}_3(p)$
4	1 or 5	$\mathrm{GSp}_4(p)$
4	2 or 4	$\mathrm{GL}_2(p) \wr C_2$
4	2 or 4	$\Gamma\mathrm{L}_2(p^2)$
4	3	$\mathrm{GL}_2(p) \otimes \mathrm{GL}_2(p)$
4	6	$\mathrm{GL}_4(p)$

UCS-groups with exponent p^2, $p > 2$

$\dim P/\Phi(P) = \dim \Phi(P)$	condition	$\mathrm{Aut}(P)$ on $P/\Phi(P)$
3		$\mathrm{SO}_3(p)$
4	$p \neq 5$	$5 \cdot 4$
4	$p \equiv \pm 2 \pmod 5$	5

So the number of isomorphism types of UCS-groups of exponent p^2 ($p > 2$) with 4 generators is

- 0, if $p = 5$;
- 1, if $p \equiv \pm 1 \pmod 5$;
- 2, if $p \equiv \pm 2 \pmod 5$.

Exterior self-quotient modules

Let P be a UCS-group of exponent p^2, $p > 2$.

$x \mapsto x^p$ is a G-module isomorphism between $V = P/P^p$ and $W = P^p$

$(x, y) \mapsto [x, y]$ can be considered as a G-module homomorphism from $V \wedge V$ onto $W = P'$

Definition. A G-module V is called an **exterior self-quotient module** (briefly **ESQ-module**) if $V \wedge V$ has a quotient module isomorphic to V. We also say that $G \leq \mathrm{GL}(V)$ is an **ESQ-group**.

If P is a UCS-group of exponent p^2 ($p > 2$), and G is the action of $\mathrm{Aut}(P)$ on $V = P/\Phi(P)$, then V is an ESQ G-module.

Research problem

There exist a UCS-group of exponent p^2 $(p > 2)$ with d generators (i.e., of order p^{2d}) if and only if there exists an irreducible subgroup $G \leq \mathrm{GL}_d(p)$ such that the natural G-module $\mathbf{F}_p^d$ is an ESQ G-module.

Problem. For which pairs d, $\mathbf{F}$ does there exist a finite irreducible group $G \leq \mathrm{GL}_d(\mathbf{F})$ such that the natural G-module is an ESQ G-module?

The Problem makes sense for any field, not just prime fields, and also for fields of characteristic 2 or 0 (although these cases have no relevance for the description of UCS p-groups).

Some examples of ESQ-modules (1)

Let $L_t = \{x \mapsto ax + b \mid a, b \in \mathbf{F}_t, a \neq 0\}$ be the group of linear functions acting as permutation matrices of degree t over a field $\mathbf{F}$, where t is a power of a prime different from the characteristic of $\mathbf{F}$. Then L acts absolutely irreducibly on the $(t-1)$-dimensional subspace of vectors with coordinate sum 0, and this is an ESQ-module, since this is the only faithful irreducible representation of L over $\mathbf{F}$.

Let q be a prime power, r a prime number, and let d be the order of q modulo r. Assume that there exist $0 < i < j < d$ such that $q^i + q^j \equiv 1 \pmod{r}$. Then any cyclic subgroup of order r in $\mathrm{GL}_d(q)$ is an ESQ-group, since the eigenvalues of a linear transformation of order r are ϵ, ϵ^q, ..., $\epsilon^{q^{d-1}}$ (where ϵ is an r-th root of unity belonging to the field of order q^d), and the eigenvalues on the exterior square are $\epsilon^{q^i+q^j}$ for $0 \leq i < j < d$.

Some examples of ESQ-modules (2)

Let $p \geq 5$ be a prime. Then the irreducible modules of $\mathrm{PSL}_2(p)$ over $\mathbf{F}_p$ are V_1, V_3, ..., V_p — one for each odd dimension up to p. We have

$$V_7 \wedge V_7 = V_3 \oplus V_7 \oplus V_{11},$$

so V_7 is an ESQ $\mathrm{PSL}_2(p)$-module if $p \geq 11$. (Actually, for $p = 7$ as well.) In fact, $\mathrm{PSL}_2(p)$ is a *minimal* irreducible ESQ-subgroup of $\mathrm{GL}_7(p)$.

Both of the two 7-dimensional irreducible representations of $G_2(2)$ over fields of characteristic different from 2 are ESQ.

Trivialities

Subgroups of ESQ-groups are ESQ-groups themselves. (But we can, of course, loose irreducibility.)

ESQ-groups remain ESQ-groups under field extensions. (But, again, irreducibility maybe lost.)

The dimension of an ESQ-module is at least 3. (Since $\dim V \wedge V = \binom{\dim V}{2}$.)

An ESQ-group cannot contain any scalar transformation except the identity. (Since λ acts as λ^2 on $V \wedge V$.)

If the eigenvalues of an element g in an ESQ-group are $\lambda_1, \ldots,$ λ_d, then there is an injective map $i \mapsto (j, k)$ ($i = 1, \ldots, d$; $1 \leq j < k \leq d$) such that $\lambda_i = \lambda_j \lambda_k$. (Since the eigenvalues of g on V must occur among the eigenvalues of g on $V \wedge V$.)

ESQ-modules of dimension 3

Let $V = \mathbf{F}^3$, then $g \in \mathrm{GL}(V)$ acts on $V \wedge V$ as $\det(g)g^{-\top}$, hence $G \leq \mathrm{GL}(V)$ is an ESQ-group iff $G \leq \mathrm{SO}(V)$.

If $\mathrm{char}(\mathbf{F}) \neq 2$, then $\mathrm{SO}_3(\mathbf{F})$ is irreducible, so there exist 3-dimensional irreducible ESQ-modules.

If $\mathrm{char}(\mathbf{F}) = 2$, then $\mathrm{SO}_3(\mathbf{F})$ is not irreducible, so there is no 3-dimensional irreducible ESQ-module in this case.

ESQ-modules of dimension 4

Theorem. If the characteristic of $\mathbf{F}$ is not 5, then the group $L \leq \mathrm{GL}_4(\mathbf{F})$ generated by the matrices

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -1 & -1 & -1 \end{bmatrix}, \qquad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

is an ESQ-group of order 20.

Conversely, if $\mathrm{char}(\mathbf{F}) \neq 2$ and $G \leq \mathrm{GL}_4(\mathbf{F})$ is a finite irreducible ESQ-group, then $\mathrm{char}(\mathbf{F}) \neq 5$, and G is conjugate to a subgroup of L, the order of G is divisible by 5, and if 5 is a square in $\mathbf{F}$, then $|G| = 20$.

ESQ-modules of dimension 5

Theorem. Let p be a prime, q a prime-power, and let G be a minimal irreducible ESQ-subgroup of $\mathrm{GL}_p(q)$. Then one of the following holds:
(a) G is not absolutely irreducible, $r = |G|$ is prime, $q^p \equiv 1 \pmod{r}$, $q \not\equiv 1 \pmod{r}$, and there exist $0 < i < j < p$ such that $q^i + q^j \equiv 1 \pmod{r}$;
(b) G is an absolutely irreducible non-abelian simple group;
(c) G is absolutely irreducible, $|G| = pr^s$, where $r \neq p$ is prime, $q \equiv 1 \pmod{r}$, G' is a minimal normal subgroup of G of order r^s.

If $p = 5$ then case (b) does not occur — using results of Di Martino and Wagner. In case (a) $|G| = 11$, in case (c) $|G| = 55$. So there exists an irreducible ESQ-subgoup of $\mathrm{GL}_5(q)$ iff $q^5 \equiv 1 \pmod{11}$.