On the irreducibility of the Dirichlet polynomial of a simple group of Lie type

Massimiliano Patassini

Università di Padova

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The Dirichlet polynomial of a group

Let G be a finite group and let N be a normal subgroup of G. The Dirichlet polynomial of G given G/N is

$$P_{G,N}(s) = \sum_{k \geq 1} \frac{a_k(G,N)}{k^s}, \quad \text{where } a_k(G,N) = \sum_{\substack{H \leq G, |G:H| = k, \\ NH = G,}} \mu_G(H).$$

Here μ_G is the Möbius function of the subgroup lattice of G, which is defined inductively by $\mu_G(G) = 1$, $\mu_G(H) = -\sum_{K > H} \mu_G(K)$.
It turns out that for each $k \in \mathbb{N}, k > 0$ the number $P_{G,N}(k)$ is the probability that k randomly chosen elements of G generate G given that they generate G/N.

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The ring of Dirichlet polynomials

Let $\mathcal{R}$ be the ring of Dirichlet finite series (also called Dirichlet polynomials) with integer coefficients, i.e.

$$\mathcal{R} = \left\{ \sum_{m \geq 1} \frac{a_m}{m^s} : a_m \in \mathbb{Z}, |\{m : a_m \neq 0\}| < \infty \right\}.$$

In particular, $\mathcal{R}$ is a factorial domain and $P_G(s)$ and $P_{G,N}(s)$ are elements of $\mathcal{R}$.

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Motivations

Questions

- Under which conditions is $P_{G,N}(s)$ irreducible?
- Can we obtain insights into the structure of the group G from some informations about how $P_G(s)$ factorizes?

Lemma

If $P_G(s)$ is irreducible, then $G/\mathrm{Frat}(G)$ is a simple group.

The converse is not true: $P_{\mathrm{PSL}_2(7)}(s)$ is reducible!

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The class $\mathcal{L}$ and the main result.

The class $\mathcal{L}$ consists of groups G satisfying the followings:

- G is a monolithic primitive group with a simple component S isomorphic to a simple group of Lie type.
- Let $X = N_G(S)/C_G(S)$. Let k be the maximum of the orders of the graph automorphisms in $X \lesssim \mathrm{Aut}(S)$. The Lie rank of S is greater than k.
- S is not isomorphic to one of the following groups: $A_2(2)$, ${}^2A_3(3^2)$, ${}^2A_4(2^2)$, ${}^2A_5(2^2)$, $A_2(p)$, $C_2(p)$ for p a Mersenne prime.

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Let G be a simple group of Lie type. The Dirichlet polynomial $P_G(s)$ is reducible in $\mathcal{R}$ if and only if $G \cong A_1(p)$ where p is a Mersenne prime such that $\log_2(p+1) \equiv 3 \pmod 4$.

The above theorem was proved by Damian, Lucchini, Morini (2004) for $G \cong A_1(p)$, p prime.

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Definitions

Let π be a set of prime numbers and let $f(s)=\sum_{m\in\mathbb{N}}\frac{a_m}{m^s}$ be a Dirichlet polynomial. We define a new Dirichlet polynomial

$$f^{(\pi)}(s)=\sum_{m\in\mathbb{N}}\frac{b_m}{m^s} \quad \text{where } b_m=\begin{cases} a_m & \text{if } m \text{ is a } \pi' \text{ number} \\ 0 & \text{otherwise.}\end{cases}$$

Example

Let $S=\mathrm{Alt}_5$. We have

$$P_S(s)=1-5^{1-s}-6^{1-s}-10^{1-s}+20^{1-s}+2\cdot 30^{1-s}-60^{1-s},$$

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Connection between monolithic primitive groups and almost simple groups.

Theorem (Jiménez-Seral, 2008)

Let G be a monolithic primitive group with a non abelian simple component S and let $X = N_G(S)/C_G(S)$. Let $n = |G : N_G(S)|$. We have

$$P^{(r)}_{G,\mathrm{Soc}(G)}(s) = P^{(r)}_{X,\mathrm{Soc}(X)}(ns - n + 1)$$

for each prime divisor r of the order of S.

In particular, if S is a simple group of Lie type of characteristic p, we have that:

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Since $P^{(p)}_{X,\mathrm{Soc}(X)}(s)$ depends on the structure of the parabolic subgroups of X,we were able to prove the following:

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Let G be in $\mathcal{L}$. Then $P^{(p)}_{G,\mathrm{Soc}(G)}(s)$ is irreducible in $\mathcal{R}$.

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A technical Lemma

If $f(s)=\sum_{m\in\mathbb{N}}\frac{a_m}{m^s}\in\mathcal{R}$ and r is a prime number, then let $|f(s)|_r=\max\{|m|_r : a_m\neq 0\}$. We call $|f(s)|_r$ the r-**part of** $f(s)$.

Lemma

Let $h(s)=\sum_{m\in\mathbb{N}}\frac{a_m}{m^s}\in\mathcal{R}$ and p be a prime. Let $m=\mathrm{lcm}\{m : a_m\neq 0\}$. Assume that

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If $f(s)=\sum_{m\in\mathbb{N}}\frac{a_m}{m^s}\in\mathcal{R}$ and r is a prime number, then let $|f(s)|_r=\max\{|m|_r : a_m\neq 0\}$. We call $|f(s)|_r$ the r-**part of** $f(s)$.

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Let $h(s)=\sum_{m\in\mathbb{N}}\frac{a_m}{m^s}\in\mathcal{R}$ and p be a prime. Let $m=\mathrm{lcm}\{m : a_m\neq 0\}$. Assume that

1. *$h^{(p)}(s)$ is irreducible;*
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The result

Proposition

If $G \in \mathcal{L}$, then $P_{G,\mathrm{Soc}(G)}(s)$ is irreducible.

We apply the previous Lemma with $h(s) = P_{G,\mathrm{Soc}(G)}(s)$. Note that m divides $|\mathrm{Soc}(G)|$.

1. $P^{(p)}_{G,\mathrm{Soc}(G)}(s)$ is irreducible.
2. Take $\pi = \pi(S) - \pi(B)$. In fact,

$$|P^{(p)}_{G,\mathrm{Soc}(G)}(s)|_r = |P^{(p)}_{X,S}(s)|^n_r = |S : B|^n_r = |S|^n_r = |\mathrm{Soc}(G)|_r$$

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Conclusions

- The result on the irreducibility of the Dirichlet polynomials $P_{G,\mathrm{Soc}(G)}(s)$ with $G \in \mathcal{L}$ can be extended.
- There are some similar results for the alternating groups: Damian, Lucchini, Morini (2004) proved that $P_{\mathrm{Alt}_p}(s)$ is irreducible for each p prime and Marilena Massa showed that $P_{\mathrm{Alt}_{p+1}}(s)$ is irreducible for each p prime.

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