On the irreducibility of the Dirichlet polynomial of a simple group of Lie type

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The Dirichlet polynomial of a group

Let G be a finite group and let N be a normal subgroup of G. The Dirichlet polynomial of G given G/N is

$$P_{G,N}(s) = \sum_{k \ge 1} \frac{a_k(G,N)}{k^s}, \text{ where } a_k(G,N) = \sum_{\substack{H \le G, |G:H| = k, \\ NH = G,}} \mu_G(H).$$

Here μ_G is the Möbius function of the subgroup lattice of G, which is defined inductively by $\mu_G(G) = 1$, $\mu_G(H) = -\sum_{K>H} \mu_G(K)$. It turns out that for each $k \in \mathbb{N}$, k > 0 the number $P_{G,N}(k)$ is the probability that k randomly chosen elements of G generate G given that they generate G/N.

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The ring of Dirichlet polynomials

Let \mathcal{R} be the ring of Dirichlet finite series (also called Dirichlet polynomials) with integer coefficients, i.e.

$$\mathcal{R} = \left\{ \sum_{m \ge 1} \frac{a_m}{m^s} : a_m \in \mathbb{Z}, |\{m : a_m \neq 0\}| < \infty \right\}.$$

In particular, $\mathcal R$ is a factorial domain and $P_G(s)$ and $P_{G,N}(s)$ are elements of $\mathcal R$.

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• Can we obtain insights into the structure of the group G from some informations about how $P_G(s)$ factorizes?

Lemma

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The simple case

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Definitions and useful theorems Irreducibility of the *p*-Dirichlet polynomial Irreducibility of $P_{G,Soc}(G)(s)$.

Definitions

Let π be a set of prime numbers and let $f(s) = \sum_{m \in \mathbb{N}} \frac{a_m}{m^s}$ be a Dirichlet polynomial. We define a new Dirichlet polynomial

$$f^{(\pi)}(s) = \sum_{m \in \mathbb{N}} \frac{b_m}{m^s}$$
 where $b_m = \begin{cases} a_m & \text{if } m \text{ is a } \pi' \text{ number} \\ 0 & \text{otherwise.} \end{cases}$

Example

Let $S = Alt_5$. We have

$$P_{S}(s) = 1 - 5^{1-s} - 6^{1-s} - 10^{1-s} + 20^{1-s} + 2 \cdot 30^{1-s} - 60^{1-s},$$

$$P_S^{(2)}(s) = 1 - 5^{1-s}, \ P_S^{(3)}(s) = 1 - 5^{1-s} - 10^{1-s} + 20^{1-s},$$

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Connection between monolithic primitive groups and almost simple groups.

Theorem (Jiménez-Seral, 2008)

Let G be a monolithic primitive group with a non abelian simple component S and let $X = N_G(S)/C_G(S)$. Let $n = |G : N_G(S)|$. We have

$$P_{G,\text{Soc}(G)}^{(r)}(s) = P_{X,\text{Soc}(X)}^{(r)}(ns - n + 1)$$

for each prime divisor r of the order of S.

In particular, if S is a simple group of Lie type of characteristic p, we have that:

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Since $P_{X,\text{Soc}(X)}^{(p)}(s)$ depends on the structure of the parabolic subgroups of X, we were able to prove the following:

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Let G be in \mathcal{L} . Then $P_{G,Soc(G)}^{(p)}(s)$ is irreducible in \mathcal{R} .

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Definitions and useful theorems Irreducibility of the *p*-Dirichlet polynomial Irreducibility of $P_{G,Soc}(G)(s)$.

A technical Lemma

If $f(s) = \sum_{m \in \mathbb{N}} \frac{a_m}{m^s} \in \mathcal{R}$ and r is a prime number, then let $|f(s)|_r = \max\{|m|_r : a_m \neq 0\}$. We call $|f(s)|_r$ the r-part of f(s).

Lemma

Let
$$h(s) = \sum_{m \in \mathbb{N}} \frac{a_m}{m^s} \in \mathcal{R}$$
 and p be a prime. Let $m = \operatorname{lcm}\{m : a_m \neq 0\}$. Assume that

h^(p)(s) is irreducible;

Output the exists Ø ≠ π ⊆ π(m) such that |h^(p)(s)|_r = |m|_r for each r ∈ π;

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$$(h(s), h^{(\pi)}(s)) = 1.$$

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The result

Proposition

If $G \in \mathcal{L}$, then $P_{G, Soc(G)}(s)$ is irreducible.

We apply the previous Lemma with $h(s) = P_{G,Soc(G)}(s)$. Note that m divides |Soc(G)|.

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$$P_{G,\text{Soc}(G)}^{(p)}(s)$$
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② Take
$$\pi=\pi(S)-\pi(B)$$
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Conclusions

- The result on the irreducibility of the Dirichlet polynomials $P_{G, \text{Soc}(G)}(s)$ with $G \in \mathcal{L}$ can be extended.
- There are some similar results for the alternating groups: Damian, Lucchini, Morini (2004) proved that P_{Alt_p}(s) is irreducible for each p prime and Marilena Massa showed that P<sub>Alt_{p+1}(s) is irreducible for each p prime.
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