Hurwitz generation of the universal covering of $\mathrm{Alt}(n)$

joint work with M. C. Tamburini

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Introduction

Definition

A finite group G is said to be Hurwitz if it can be generated by two elements x, y such that $o(x) = 2$, $o(y) = 3$ and $o(xy) = 7$.

Equivalently, G is a finite non-trivial quotient of the infinite triangle group

$$T(2,3,7) = \langle X, Y \mid X^2 = Y^3 = (XY)^7 = 1 \rangle$$

Literature

Here, a short (and not complete) list of known results about Hurwitz groups:

- $\mathrm{Alt}(n)$: Conder (1980);
- $PSL(2, q)$: Macbeath (1969);
- $SL(n, q)$ for $n \geq 287$: Lucchini, Tamburini, Wilson (2000); for intermediate rank: Vsemirnov (2004).
- $Sp(2m, q)$, $SU(2m, q)$, $\Omega^+(2m, q)$, $SU(2m + 7, q)$, $\Omega(2m + 7, q)$ for $n \geq 371$: Lucchini, Tamburini (1999).
- $G_2(q)$, ${}^2G_2(3^{2m+1})$, ${}^3D_4(q)$, ${}^2F_4(2^{2m+1})$: Malle (1990,1995).
- It is completely known which sporadic groups are Hurwitz: Woldar (1990), Linton (1989), Wilson (2001) et al.

The universal covering $\widetilde{\mathrm{Alt}(n)}$

Notation

We denote by $\widetilde{\mathrm{Alt}(n)}$ the universal covering of the alternating group $\mathrm{Alt}(n)$.

To our purposes it is enough to recall that for $n \geq 5$, $\widetilde{\mathrm{Alt}(n)}$ is a perfect group such that

$$\frac{\widetilde{\mathrm{Alt}(n)}}{\widetilde{Z}} \cong \mathrm{Alt}(n),$$

where its center $\widetilde{Z}$ has order

$$|\widetilde{Z}| = \begin{cases} 6 & \text{for } n = 6, 7 \\ 2 & \text{otherwise} \end{cases}$$

Reduction to $\mathrm{Alt}(n)$

Theorem (cfr. Aschbacher FGT)

Let $\tilde{x}$ be a 2-element in the universal covering $\widetilde{\mathrm{Alt}(n)}$ whose image in $\mathrm{Alt}(n)$ is an involution. Then

- *$\tilde{x}$ has order 2, if x is the product of $4k$ transpositions;*
- *$\tilde{x}$ has order 4, if x is the product of $4k+2$ transpositions.*

Corollary

The group $\widetilde{\mathrm{Alt}(n)}$ is Hurwitz if and only if $\mathrm{Alt}(n)$ admits a $(2,3,7)$-generating triple (x, y, xy) in which x is the product of $4k$ transpositions.

The Alternating groups

Theorem (Conder)

The group $\mathrm{Alt}(n)$ *is Hurwitz for all* $n > 167$ *and for all the following values of* n*:*

		15						21	22		
			28	29						35	36
37					42	43		45			
49	50	51	52				56	57	58		
		63	64	65	66				70	71	72
73				77	78	79	80	81			84
85	86	87	88			91	92	93	94		96
	98	99	100	101	102			105	106	107	108
109			112	113	114	115	116	117		119	120
121	122	123	124		126	127	128	129	130		132
133	134	135	136	137	138		140	141	142	143	144
145		147	148	149	150	151	152	153	154	155	156
157	158	159	160	161	162	163	164	165	166		

A necessary condition

Genus formula:

$$n = 84(g-1) + 21r + 28s + 36t$$

where

- $g \geq 0$;
- r : number of points fixed by x;
- s : number of points fixed by y;
- t : number of points fixed by xy;

Example

For $n = 22$, the Genus formula gives as only possibility:

$$22 = 84(0-1) + 21 \cdot 2 + 28 \cdot 1 + 36 \cdot 1.$$

A necessary condition

Scott's formula:

$$4\left[\frac{n}{8}\right] + 2\left[\frac{n}{3}\right] + 6\left[\frac{n}{7}\right] \geq 2n - 2$$

Example

For $n = 22$, we have

$$4\left[\frac{22}{8}\right] + 2\left[\frac{22}{3}\right] + 6\left[\frac{22}{7}\right] = 40 < 2 \cdot 22 - 2 = 42.$$

The values $n = 8, 9, 16$ and 24 satisfy Scott's formula. However, $\mathrm{Alt}(n)$ is not Hurwitz!

A necessary condition

The values of n for which $\mathrm{Alt}(n)$ is Hurwitz, but do not satisfy the genus (or Scott's) formula are:

15	22	29	37	45	52	71	79	86	87
94	101	102	109	116	117	124	132	143	151
158	159	166	173	174	181	188	215	223	230

The exception $n = 21$

Lemma

The covering $\widetilde{\mathrm{Alt}(21)}$ is not Hurwitz.

By contradiction let (x, y, xy) be the image in $\mathrm{Alt}(n)$ of a $(2, 3, 7)$-generating triple of $\widetilde{\mathrm{Alt}(21)}$. It follows that x fixes 5 points. Scott's formula for the diagonal action of $H = \langle x, y \rangle$ on the symmetric square S of V, the irreducible component of the natural module $\mathbb{C}^{21}$, gives

$$d_S^x + d_S^y + d_S^{xy} \leq \frac{20 \cdot 21}{2} + 2.$$

d_S^h is the dimension of the space of points fixed by h. On the other hand we have $d_S^x = 114$, $d_S^y \geq 70$ and $d_S^z \geq 30$, whence the contradiction $114 + 70 + 30 = 214 \leq 212$.

The covering group $\widetilde{\mathrm{Alt}(n)}$

Theorem (P., Tamburini, 2010)

The universal covering $\widetilde{\mathrm{Alt}(n)}$ of an alternating group $\mathrm{Alt}(n)$ which is Hurwitz is still Hurwitz, except for the 31 previous exceptions.

The exceptional cases are the 30 values that do not satisfy the genus formula and $n = 21$.

Let $T(2,3,7) = \langle X, Y \mid X^2 = Y^3 = (XY)^7 = 1 \rangle$ be the infinite triangle group. Following the Conder's paper, a permutation representation of T

$$\mu : T(2,3,7) \to \mathrm{Alt}(m)$$

is representated by a diagram M with m vertices.
Set $x = \mu(X)$ and $y = \mu(Y)$.

Definition

Two vertices $j \neq k$ of M form an (i)-handle $(1 \leq i \leq 6)$ if

- x fixes both j and k;
- $(xy)^i$ sends j into k.

Diagrams

If

- μ is a representation associated to a diagram M with m vertices and (i)-handle $\{j, k\}$;
- μ' is a representation associated to a diagram M' with m' vertices and (i)-handle $\{j', k'\}$

then, we obtain a new representation

$$\Psi : T(2,3,7) \to \mathrm{Alt}(m + m')$$

extending the action

$$\begin{array}{rcl} X & \mapsto & \mu(X)\mu'(X)(j, j')(k, k') \\ Y & \mapsto & \mu(Y)\mu'(Y) \end{array}$$

Diagrams

The diagram associated to new representation Ψ is denoted by $M(i)M'$.

Let $\tau(x)$ be the number of disjoint transpositions of x appearing in its cycle decomposition. Note that

$$\tau(\Psi(X)) = \tau(\mu(X)) + \tau(\mu'(X)) + 2.$$

List of diagrams (Conder + Vsemirnov):

Diagram	deg	τ	p	Diagram	deg	τ	p
A	14	6		C	21	8	
B	15	6		E	28	12	
D	22	10		G'	42	20	
G	42	18		H_2	142	68	23
H_0	42	18	17	H_3	115	56	17
H_1	57	26	5	H_5	187	92	43
H_4	144	70	17	H_7	77	36	17
H_6	216	106	5	H_8	36	16	5
H_{10}	136	66	5	H_9	135	64	19
J	72	34		H_{11}	165	80	19
O	7	2		H_{12}	180	88	47
P	15	6		H_{13}	195	96	23
R	22	10		Q	21	8	
				S	36	16	
				T	66	32	

A new diagram

Lemma

Let x and y be defined by the diagram G, with vertices $\{1, \ldots, 42\}$ and (1)-handles $\{2, 3\}$, $\{14, 15\}$ $\{32, 33\}$. Set

$$x' = x(14, 32)(15, 33).$$

Then $x'y$ and $[x', y]$ are respectively conjugate to xy and $[x, y]$

We call G' this new diagram: it is of degree 42, with a (1)-handle $\{2, 3\}$ and x' is the product of 20 transpositions.

First example: $n = 175$

1. $175 \equiv 7 \pmod{14}$.
   So, take the diagram H_7 of degree 77.

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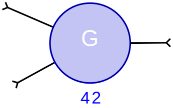
   So, take the diagram H_7 of degree 77.
2. $175 = 77 + 2 \cdot 42 + 14$.
   So, take 2 copies of the diagram G and a diagram A (of degree 14).
3. Construct the associated diagram.

First example: $n = 175$

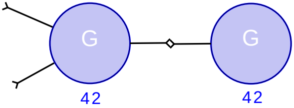
- Decompose n: $175 = 77 + 2 \cdot 42 + 14$.
- Consider the associated diagram $\begin{matrix} A(1) \\ H_7(1) \end{matrix} G(1)G$.



- $\tau(x) = 18$.

First example: $n = 175$

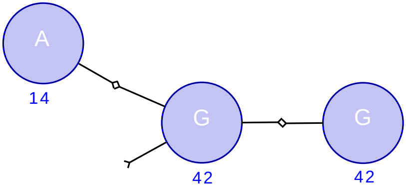
- Decompose n: $175 = 77 + 2 \cdot 42 + 14$.
- Consider the associated diagram $\substack{A(1) \\ H_7(1)} \, G(1)G$.



- $\tau(x) = 18 + 2 + 18$.

First example: $n = 175$

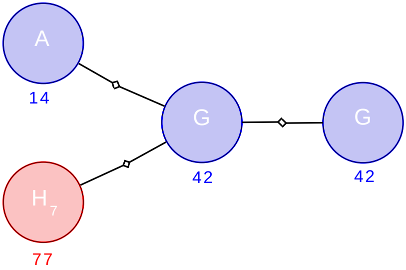
- Decompose n: $175 = 77 + 2 \cdot 42 + 14$.
- Consider the associated diagram $\begin{matrix} A(1) \\ H_7(1) \end{matrix} G(1)G$.



- $\tau(x) = 18 + 2 + 18 + 2 + 6$.

First example: $n = 175$

- Decompose n: $175 = 77 + 2 \cdot 42 + 14$.
- Consider the associated diagram $\genfrac{}{}{0pt}{}{A(1)}{H_7(1)} G(1)G$.



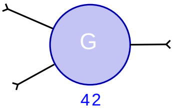
- $\tau(x) = 18 + 2 + 18 + 2 + 6 + 2 + 36 = 84$.

Second example: $n = 148$

1. $148 \equiv 8 \pmod{14}$.
   So, take the diagram H_8 of degree 36.
2. $148 = 36 + 2 \cdot 42 + 28$.
   So, take 2 copies of the diagram G and a diagram E (of degree 28).
3. Construct the diagram $\begin{smallmatrix} E(1) \\ H_8(1) \end{smallmatrix} G(1)G$.

Second example: $n = 148$

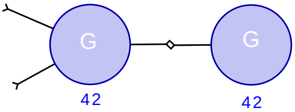
- Decompose n: $148 = 36 + 2 \cdot 42 + 28$.
- Consider the associated diagram $\begin{matrix} E(1) \\ H_8(1) \end{matrix} G(1)G$.



- $\tau(x) = 18$.

Second example: $n = 148$

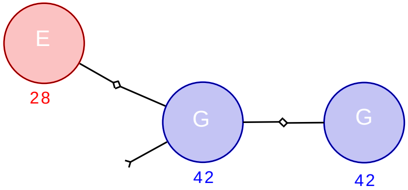
- Decompose n: $148 = 36 + 2 \cdot 42 + 28$.
- Consider the associated diagram $\begin{matrix} E(1) \\ H_8(1) \end{matrix} G(1)G$.



- $\tau(x) = 18 + 2 + 18$.

SECOND EXAMPLE: $n = 148$

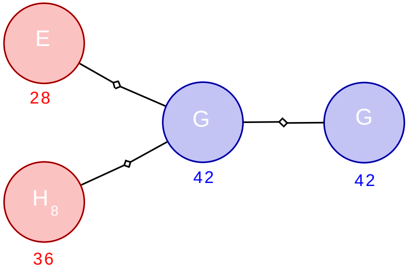
- Decompose n: $148 = 36 + 2 \cdot 42 + 28$.
- Consider the associated diagram $\begin{smallmatrix} E(1) \\ H_8(1) \end{smallmatrix} G(1)G$.



- $\tau(x) = 18 + 2 + 18 + 2 + 12$.

Second example: $n = 148$

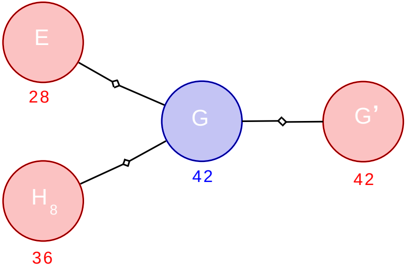
- Decompose n: $148 = 36 + 2 \cdot 42 + 28$.
- Consider the associated diagram $\begin{matrix} E(1) \\ H_8(1) \end{matrix} G(1)G$.



- $\tau(x) = 18 + 2 + 18 + 2 + 12 + 2 + 16 = 70$.

SECOND EXAMPLE: $n = 148$

- Decompose n: $148 = 36 + 2 \cdot 42 + 28$.
- Consider the associated diagram $\begin{matrix} E(1) \\ H_8(1) \end{matrix} G(1)G$.



- $\tau(x) = 18 + 2 + \mathbf{20} + 2 + 12 + 2 + 16 = 72$.

The general case

Consider the values of n of shapes:

$$\begin{aligned} n &= 42 + d \\ n &= 42r + 14s + d,\ r \geq 2, s = 0, 1, 2, \end{aligned}$$

where d is the degree of the unique diagram H_i such that

$$n \equiv i \pmod{14}.$$

For each diagram H_i, we define three composite diagrams, namely:

$$\Omega_0^i = H_i(1)G, \quad \Omega_1^i = \begin{smallmatrix} H_i(1) \\ A(1) \end{smallmatrix} G, \quad \Omega_2^i = \begin{smallmatrix} H_i(1) \\ E(1) \end{smallmatrix} G.$$

The general case

For any such n, there exists a composite diagram of degree n: e.g. one of the diagrams Ω_0^i or

$$\Omega_s^i(1)\underbrace{G(1)\ldots(1)G}_{r-1 \text{ times}}, \quad s := 0, 1, 2.$$

The general case

For any such n, there exists a composite diagram of degree n: e.g. one of the diagrams Ω_0^i or

$$\Omega_s^i(1)\underbrace{G(1)\ldots(1)G}_{r-1 \text{ times}}, \quad s := 0,1,2.$$

If x and y are defined by these diagrams, then $\langle x, y\rangle$ is a primitive subgroup of $\mathrm{Alt}(n)$ containing a p-cycle of prime length $p \leq n-3$. By a result of Jordan, $\langle x, y\rangle = \mathrm{Alt}(n)$. In this case, the p-cycle is a power of the commutator $[x, y]$.

THE GENERAL CASE

If $\tau(x) \equiv 0 \pmod 4$, the group $\widetilde{\mathrm{Alt}(n)}$ is Hurwitz.
Otherwise we may consider the diagram obtained substituting the last copy of G by G'.

The number of 2-cycles of the involution x' defined by this modified diagram is $\tau(x') = \tau(x) + 2 \equiv 0 \pmod 4$.

Since the cycle structure of $[x', y]$ is the same of $[x, y]$, we conclude that $\langle x', y \rangle = \mathrm{Alt}(n)$ by the same argument used for $\langle x, y \rangle$.

The general case

Note that every $n \geq 300$ and the 106 values listed below have right shape:

78	84	99	119	120	126	134	140	141	148	154
155	157	161	162	168	169	175	176	177	178	182
183	184	186	189	190	196	197	199	203	204	207
210	211	213	217	218	219	220	222	224	225	226
227	228	229	231	232	233	234	237	238	239	240
241	242	245	246	247	248	249	252	253	254	255
256	258	259	260	261	262	263	264	266	267	268
269	270	271	273	274	275	276	277	278	279	280
281	282	283	284	285	287	288	289	290	291	292
293	294	295	296	297	298	299				

Small cases

To solve the remaining cases ($n < 300$), we consider the diagrams:

$H_{i_1}(1)E$	H_{i_2}	$O(1)H_{i_2}$	$A(1)H_{i_2}$
$R(1)H_{i_2}$	$\begin{matrix} H_{i_1}(1) \\ E(1) \end{matrix} G$	$\begin{matrix} H_{i_2}(1) \\ A(1) \end{matrix} G$	$P(1)G(1)H_{i_2}$
$\begin{matrix} A(1) \\ A(1) \end{matrix} G(1)H_{i_2}$	$P(1)H_3$	$P(1)H_9$	$\begin{matrix} R(1) \\ H_8(1) \end{matrix} G$
$\begin{matrix} A(1) \\ P(1) \end{matrix} G(1)H_8$	$\begin{matrix} A(1) \\ R(1) \end{matrix} G(1)H_8$	$\begin{matrix} P(1) \\ P(1) \end{matrix} G(1)H_8$	

Low cases

For each of these diagrams, a suitable power of the commutator $[x, y]$ is the p-cycle listed before, associated to the H_i involved by the diagram. They provide the following 59 values of n

36	43	50	58	70	77	85	91	92	93	100
106	107	108	112	114	115	122	127	129	130	133
135	137	142	147	149	150	156	164	165	171	172
179	180	185	187	191	192	194	195	198	201	202
205	206	209	212	214	221	235	236	243	244	250
251	257	265	286							

Low cases

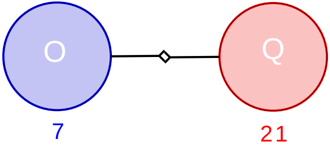
For the following 35 values of n we take a suitable diagram, giving explicitely the right prime p and the p-cycle.

$n =$

28	35	42	49	51	57	63	64	65	66	72
73	80	81	88	98	113	123	138	105	121	128
136	144	145	152	153	160	163	170	193	200	208
216	272									

First example: $n = 28$

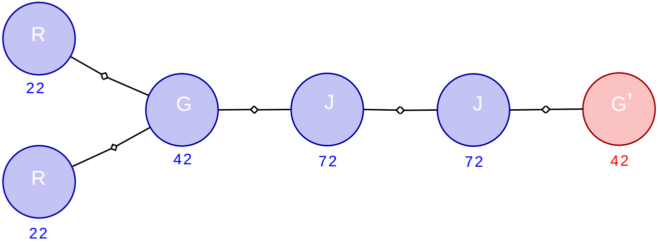
- Decompose n: $n = 7 + 21$.
- Consider the associated diagram $O(1)Q$.



- $\tau(x) = 2 + 2 + 8 = 12$.
- 13-cycle: $(xy^2xyxyxy^2)^{24}$.

SECOND EXAMPLE: $n = 272$

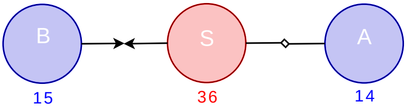
- Decompose n: $n = 2 \cdot (22 + 42 + 72)$.
- Consider the associated diagram $\substack{R(1) \\ R(1)} G(1)J(1)J(1)G'$.



- $\tau(x) = 10 + 2 + 10 + 2 + 18 + 2 + 34 + 2 + 34 + 2 + 20 = 100$.
- 17-cycle: $(xyxy^2xyxyxy^2xyxy^2xy^2xy^2xy)^{155610}$.

Third example: $n = 65$

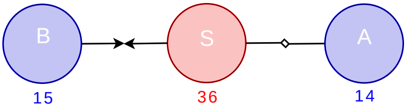
- Decompose n: $65 = 15 + 36 + 14$.
- Consider the associated diagram $B(2)S(1)A$.



- $\tau(x) = 6 + 2 + 16 + 2 + 6 = 32$.

THIRD EXAMPLE: $n = 65$

- Decompose n: $65 = 15 + 36 + 14$.
- Consider the associated diagram $B(2)S(1)A$.



- $\tau(x) = 6 + 2 + 16 + 2 + 6 = 32$.
- The same may applied to the case $n = 72$, considering the diagram $D(2)S(1)A$.

The last two cases

$n = 56$

$$\begin{aligned} x &= (1,52)(2,6)(3,7)(4,53)(5,9)(8,12)(10,15)(11,13)(14,18) \\ &\quad (16,21)(17,22)(19,24)(20,34)(23,27)(25,30)(26,32) \\ &\quad (28,33)(29,41)(31,36)(35,54)(37,42)(38,40)(39,45) \\ &\quad (43,48)(44,49)(46,51)(47,56)(50,55) \\ y &= \prod_{i=0}^{16}(3i+1, 3i+2, 3i+3). \end{aligned}$$

41-cycle: $(xyxyxy^2xy^2xyxy^2xyxy^2xy^2)^{13}$.

The last two cases

$n = 96$

$$
\begin{aligned}
x &= (1,2)(3,4)(5,7)(6,10)(8,13)(9,16)(11,19)(12,14)(15,22)\\
&\quad (17,25)(18,28)(20,23)(21,31)(24,30)(26,34)(27,37)(29,35)\\
&\quad (32,33)(36,40)(38,43)(39,46)(41,48)(42,49)(44,52)(45,55)\\
&\quad (47,58)(54,61)(57,64)(59,67)(60,70)(62,63)(65,72)(66,68)\\
&\quad (69,73)(71,76)(50,56)(51,53)(74,79)(75,82)(77,85)(78,88)\\
&\quad (80,90)(81,91)(83,89)(84,86)(87,94)(92,93)(95,96).\\
y &= \prod_{i=0}^{31}(3i+1, 3i+2, 3i+3).
\end{aligned}
$$

59-cycle: $(xyxy^2xyxyxy^2xyxy^2)^{420}$.

Alt(n) IS HURWITZ IF AND ONLY IF $n =$

		15						21	22		
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37					42	43		45			
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	98	99	100	101	102			105	106	107	108
109			112	113	114	115	116	117		119	120
121	122	123	124		126	127	128	129	130		132
133	134	135	136	137	138		140	141	142	143	144
145		147	148	149	150	151	152	153	154	155	156
157	158	159	160	161	162	163	164	165	166		168
169	170	171	172	173	174	175	176	177	178	179	180
181	182	183	184	185	186	187	188	189	190	191	192
193	194	195	196	197	198	199	200	201	202	203	204
205	206	207	208	209	210	211	212	213	214	215	216
217	218	219	220	221	222	223	224	225	226	227	228
229	230	231	232	233	234	235	...	...	...	...	

$\widetilde{\mathrm{Alt}(n)}$ is Hurwitz if and only if $n =$

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109			112	113	114	115	116	117		119	120
121	122	123	124		126	127	128	129	130		132
133	134	135	136	137	138		140	141	142	143	144
145		147	148	149	150	151	152	153	154	155	156
157	158	159	160	161	162	163	164	165	166		168
169	170	171	172	173	174	175	176	177	178	179	180
181	182	183	184	185	186	187	188	189	190	191	192
193	194	195	196	197	198	199	200	201	202	203	204
205	206	207	208	209	210	211	212	213	214	215	216
217	218	219	220	221	222	223	224	225	226	227	228
229	230	231	232	233	234	235	...	...	...	...	

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85			88			91	92	93			96
	98	99	100					105	106	107	108
			112	113	114	115				119	120
121	122	123			126	127	128	129	130		
133	134	135	136	137	138		140	141	142		144
145		147	148	149	150		152	153	154	155	156
157			160	161	162	163	164	165			168
169	170	171	172			175	176	177	178	179	180
	182	183	184	185	186	187		189	190	191	192
193	194	195	196	197	198	199	200	201	202	203	204
205	206	207	208	209	210	211	212	213	214		216
217	218	219	220	221	222		224	225	226	227	228
229		231	232	233	234	235	...	...	...	...	