Sylow Permutability in Locally Finite Groups

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1. Permutability and Sylow Permutability

A subgroup H of a group G is *permutable* if $HK = KH$ for all subgroups K of G: denote this by

$$H \textit{ per } G.$$

Also H is said to be *Sylow permutable* (or *S-permutable*) if $HP = PH$ for all Sylow subgroups P of G, in symbols

$$H \textit{ S-per } G.$$

We recall some basic results.

(i) If H *per* G and G is finite, then H is subnormal in G, (i.e., H *sn* G). (O. Ore [O]).

(ii) If H *per* G for any group G, then H is ascendant in G, (H *asc* G). (S. Stonehewer [S]).

(iii) **Theorem.** *If* H S*-per* G *and* G *is finite, then* H *sn* G. (O. Kegel [K]).

Proof. It is enough to show that if H is a maximal (proper) S-permutable subgroup of G, then $H \lhd G$. Assume this is false and G is a smallest counterexample. Then there is a $P \in \mathrm{Syl}_p(G)$ such that $H^P \neq H$. Choose $x \in P$ such that $H^x \neq H$. Then $H \neq J =< H, H^x >$ and J is S-permutable in G, so $J = G$ and hence $G = HP$.

Let q be a prime different from p and let $Q \in \mathrm{Syl}_q(G)$. Then

$$H \leq HQ \leq HP = G$$

and $|HQ : H|$ divides $|HP : H|$. Since $p \neq q$, we deduce that $Q \leq H$. Thus the subgroup L generated by all the p'-elements of G is contained in H. Then

$L \lhd G$ and G/L is a finite p-group, so $H\ sn\ G$. By maximality $H = H^G$ and $H \lhd G$.

Infinite groups

If H S-*per* G and G is infinite, what can be said about H? In general nothing, eg., if G is torsion-free or a p-group. We therefore restrict attention to locally finite groups.

Recall that a subgroup H of a group G is called *serial* if there is a series of general order type between H and G. Write this

$$H\ ser\ G.$$

The following criterion for seriality in locally finite groups is due to B. Hartley.

Theorem.([H]). *Let G be a locally finite group and let H be a subgroup such that $H \cap F$ sn F for every finite subgroup F. Then H is serial in G.*

Note the consequence that in a locally finite p-group G every subgroup H is serial in G. Also every subgroup of G is S-permutable. This suggests that we reformulate the question as follows.

Problem

If G is a locally finite group and H is an S-permutable subgroup, is H ser G?

This appears to be a hard problem. One difficulty is that if G is locally finite and $N \lhd G$, not every Sylow p-subgroup of G/N has the form PN/N where P is a Sylow p-subgroup of G.

The statement is true in special situations.

(i) *If H is a finite S-permutable subgroup of a locally finite group G, then H ser G.*

Proof. Let F be a finite subgroup of G and put $J =< H, F >$, which is finite. Now H S-*per* G implies that H S-*per* J, so H *sn* J by Kegel's theorem. Hence $H \cap F$ *sn* $J \cap F = H$. It follows by Hartley's theorem that H *ser* G.

Note: we cannot conclude that H *asc* G since there are locally finite p-groups with no non-trivial ascendant cyclic subgroups.

The next result is due to A. Ballester-Bolinches, L. Kurdachenko, J. Otal and T. Pedraza [BKOP2].

Theorem. *If H is an S-permutable subgroup of a hyperfinite group G, then H asc G.*

Sketch of Proof. Let $\{G_\alpha \mid \alpha \leq \beta\}$ be an ascending normal series with finite factors. Then H S*–per* HG_1 and G_1 is finite. Put $C = \mathrm{Core}_{HG_1}(H)$. Then HG_1/C is finite and H/C S-*per* HG_1/C. Hence H *sn* HG_1 by Kegel's theorem. Next argue that H *asc* HG_α for all α by transfinite induction, which completes the proof.

The following is a recent result on the problem.

Theorem 1. *Let G be a locally finite group satisfying min-p for all primes p. If H is S-permutable in G, then H is ascendant in G.*

Proof. Consider first the case where G has finite Sylow subgroups. The key step is to show that if H is a proper S-permutable subgroup, there exists an S-permutable subgroup K properly containing H such that $H \triangleleft K$.

To see this assume that H is not normal in G and find $P \in \mathrm{Syl}_G(p)$ such that

$H^P \neq H$. Since P is finite, $H \ sn \ HP$. Form the series of successive normal closures of H in $J = HP$

$$H = H_k \lhd H_{k-1} \lhd \cdots \lhd H_1 \lhd H_0 = J.$$

Here $k \geq 2$. Put $K = H_{k-1}$; then $K = H^{H_{k-2}}$ is S-permutable in G and $H \lhd K$. Applying this result repeatedly, we find that $H \ asc \ G$.

Now consider the general case where each Sylow subgroup of G is a Černikov group. There is a unique maximal divisible abelian subgroup D and G/D has finite Sylow subgroups. It is easy to see that HD/D *S-per* G/D and therefore HD *asc* G.

For a fixed prime p put

$$D(i) = \{x \mid x \in D, x^{p^i} = 1\}.$$

Thus $D(i) \lhd G$ is finite and H *sn* $HD(i)$ by the normal core argument. Hence

$$HD(i-1) \ sn \ HD(i)$$

for $i = 1, 2. \ldots,$ and H *asc* HD_p for all p.

Finally, if $p_1, p_2, \ldots$ is the sequence of primes, we have

$$H \ asc \ HD(p_1) \ asc \ HD(p_1)D(p_2) \ asc \ldots,$$

so $H \ asc \ HD$ and $H \ asc \ G$.

Another recent result is:

Theorem 2. *Let G be a periodic soluble group and let H be S-permutable in G. If H satisfies min-p for all primes p, then H is serial in G.*

The soluble case is still open.

2. Transitivity

A group G is called a *PT-group* if permutability is transitive in G, i.e.,

$$H \ per \ K \ per \ G \Rightarrow H \ per \ G.$$

Also G is called a $PST - group$ if S-permutability is transitive, i.e.,

$$H \ S\text{-}per \ K \ S\text{-}per \ G \Rightarrow H \ S\text{-}per \ G$$

The following remarks are consequences of the theorems of Ore and Kegel.

(i) *A finite group is a PT-group if and only if the subnormal subgroups and the permutable subgroups are the same.*

(ii) *A finite group is a PST-group if and only if the subnormal subgroups and the S-permutable subgroups are the same.*

Finite PT-groups and PST-groups have been studied intensively, especially in the soluble case. There are two principal structure theorems.

Theorem. (R. Agrawal [A]). *Let G be a finite group. Then G is a soluble PST-group if and only if there is an abelian normal subgroup L of odd order such that:*

(i) G/L is nilpotent;

(ii) $\pi(L) \cap \pi(G/L) = \emptyset$;

(iii) elements of G induce power automorphisms in L.

Theorem. (G. Zacher [Z]). *Let G be a finite group. Then G is a soluble PT-group if and only if there is an abelian normal subgroup L of odd order such that:*

(i) G/L is nilpotent with modular Sylow subgroups, (i.e., it is an Iwasawa group);

(ii) $\pi(L) \cap \pi(G/L) = \emptyset$;

(iii) elements of G induce power automorphisms in L.

Call these *theorems of Gaschütz type*, after W. Gaschütz's classification of finite solluble T-groups [G]. So the difference between finite soluble PT-groups and PST-groups is that the former have modular Sylow subgroups.

There are also structure theorems for finite insoluble T-, PT-, PST-groups, which depend on the Schreier Conjecture and hence on the classification of finite simple groups – see [R2].

3. Locally finite PT- and PST-groups

If we want to construct a theory of locally finite PST- and PT-groups, there are significant obstacles:

(i) lack of a good Sylow theory, eg., failure of conjugacy of Sylow subgroups;

(ii) falsity of the Schreier Conjecture for infinite simple groups.

Some results are known in special cases.

(a) Periodic soluble T- and PT-groups ([R1],[M]). But periodic soluble PST-groups have not been studied.

(b) Hyperfinite, radical PST-groups: necessary conditions are known ([BKOP2]).

Here are recent results for locally finite groups whose finite subgroups are PT-groups or PST -groups ([R4]).

Theorem 3. *Let G be a locally finite group. Then the following conditions are equivalent.*

(a) Finite subgroups of G are PST-groups.

(b) There is an abelian normal subgroup L containing no involutions such that G/L is locally nilpotent, $\pi(L) \cap \pi(G/L) = \emptyset$ and elements of G induce power automorphisms in L.

(c) In each section of G the S-permutable subgroups and the serial subgroups are

the same.

(d) Each section of G is a PST-group.

Theorem 4. *Let G be a locally finite group. Then the following conditions are equivalent.*

(a) Finite subgroups of G are PT-groups.

(b) There is an abelian normal subgroup L containing no involutions such that G/L is an Iwawasa group, $\pi(L) \cap \pi(G/L) = \emptyset$ and elements of G induce power automorphisms in L.

(c) In each section of G the S-permutable

subgroups and the ascendant subgroups are the same.

(d) Each section of G is a PT-group.

Notice that the conditions (b) are of Gaschütz type. Also Theorems 3 and 4 are a type of local theorem for PST and PT.

Corollary. *A locally finite minimal non-PST-group or PT-group is finite.*

This fills a gap in the proofs in [R3], where the finite minimal non-PST and PT-groups are classified.

The main step in the proof of Theorem

3 is (b) $\Rightarrow$ (c); it uses the following fact.

Lemma. *Let $G = AB$ be a periodic group where A, B are abelian p-groups. Then G is a locally finite p-group.*

This result is a consequence of a theorem of Sysak and Černikov (see [AFG]); it also follows from a result in [RS]:

Theorem. *Let $G = AB$ where A, B are abelian groups. Then each chief factor of G is centralized by either A or B.*

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