# Group identities for symmetric units of group rings

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# Classical and $K$-linear involutions

Let $G$ be a group endowed with an involution $\star$.
Let us consider the $K$-linear extension of $\star$ to $KG$ by setting

$$\Big(\sum_{g\in G} a_g g\Big)^\star := \sum_{g\in G} a_g g^\star.$$

This extension, which we denote again by $\star$, is an involution of $KG$ wich fixes the ground field $K$ elementwise.

## Definition

Let $KG$ be the group algebra of a group $G$ over a field $K$. If $G$ is endowed with an involution $\star$, its linear extension to the group algebra $KG$ is called a *$K$-linear involution* of $KG$. In particular, if $\star : g \mapsto g^{-1}$, $g \in G$, the induced involution is called *the classical involution*.

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$$KG^+ := \{x | \ x \in KG \quad x^\star = x\},$$

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$KG^-$ is a Lie subalgebra of $KG$, whereas $KG^+$ is a Jordan subalgebra of $KG$.

**Definition**

Let $KG$ be the group algebra of a group $G$ over a field $K$ endowed with a $K$-linear involution. The elements of $KG^+$ are called the *symmetric elements* of $KG$ (with respect to $\star$) and those of $KG^-$ are called the *skew-symmetric elements* of $KG$.

# Symmetric units

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In general, $\mathcal{U}^+(KG)$ is a subset of $\mathcal{U}(KG)$.

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If $A$ is an arbitrary algebra with involution, let us define in the same manner $A^+$ and $A^-$.

**Theorem [Amitsur,1968]**

Let $A$ be an algebra with involution. If $A^+$ or $A^-$ satisfies a polynomial identity, then $A$ satisfies a polynomial identity.

# Main question for symmetric units

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**Second Question**

To determine the extent to which the properties of the symmetric units determine the properties of the whole unit group of the group ring.

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- According to the result by Passi-Passmann-Sehgal (1973) and Khripta (1972)

$$KG \text{ is Lie nilpotent } \iff \mathcal{U}(KG) \text{ is nilpotent;}$$

- According to the result by Passi-Passmann-Sehgal (1973) and A. Bovdi (2005) (under “small”' restrictions on $K$ and $G$)

$$KG \text{ is Lie solvable } \iff \mathcal{U}(KG) \text{ is solvable;}$$

- According to the result by Sehgal (1978) and A. Bovdi (2006)

$$KG \text{ is Lie } n\text{-Engel } \iff \mathcal{U}(KG) \text{ is } m\text{-Engel.}$$

In this spirit is the following

**Third Question**

Do the Lie identities satisfied by the symmetric elements reflect the group identities satisfied by the symmetric units of the group ring?

After the papers by Giambruno, Jespers, Sehgal, Valenti and Liu which solved the

### Hartley's Conjecture

Assume that $K$ is a field and $G$ is a torsion group, then

$$\mathcal{U}(KG) \text{ is } GI \implies KG \text{ is } PI.$$

and those by Liu, Passman, Giambruno, Sehgal and Valenti which classified when $\mathcal{U}(KG)$ is GI, the attention moved to the symmetric units with respect to the classical involution.

# When $\mathcal{U}^+(KG)$ is GI

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## Theorem [Giambruno-Sehgal-Valenti, 1998]

Let $KG$ be the group algebra of a torsion group $G$ over an infinite field $K$ of characteristic $p \neq 2$.

(a) If $p = 0$, $\mathcal{U}^+(KG)$ is GI if, and only if, $G$ is either abelian or Hamiltonian 2-group.

(b) If $p > 2$, $\mathcal{U}^+(KG)$ is GI if, and only if, $KG$ is PI and either $Q_8 \not\subseteq G$ and $G'$ is of bounded exponent $p^k$ for some $k \geq 0$ or $Q_8 \subseteq G$ and

- $P$ is a normal subgroup of $G$ and $G/P$ is a Hamiltonian 2-group;
- $G$ is of bounded exponent $4p^s$ for some $s \geq 0$.

# When $\mathcal{U}^+(KG)$ is GI

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- Sehgal-Valenti [*Group algebras with symmetric units satisfying a group identity.* Manuscripta Math. **119** (2006), 243-254] studied the non-torsion case.

# Special group identities

After the result by Giambruno, Sehgal and Valenti, it was of interest to consider when $\mathcal{U}^+(KG)$ satisfies special group identities.

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**Theorem [Lee, 2003]**

Let $KG$ be the group algebra of a torsion group $G$ over a field $K$ of characteristic $p > 2$ endowed with the classical involution. Then $\mathcal{U}^+(KG)$ is nilpotent if, and only if, one of the following statements occurs:

(i) $Q_8 \not\subseteq G$ and $G$ is nilpotent and $p$-abelian;

(ii) $Q_8 \subseteq G$ and $G \cong Q_8 \times E \times P$, where $E^2 = 1$ and $P$ is a finite $p$-group.

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- Lee-Polcino Milies-Sehgal [*Group rings whose symmetric units are nilpotent.* J. Group Theory **10** (2007), 685-701] studied the non-torsion case.

By virtue of the results by Passi-Passman-Sehgal (1973) and Khripta (1972) one has

## Corollary 1

Let $KG$ be the group algebra of a torsion group $G$ such that $Q_8 \not\subseteq G$ over a field $K$ of characteristic $p \neq 2$ endowed with the classical involution. The following statements are equivalent:

(i) $\mathcal{U}^+(KG)$ is nilpotent;

(ii) $KG^+$ is Lie nilpotent;

(iii) $KG$ is Lie nilpotent;

(iv) $\mathcal{U}(KG)$ is nilpotent;

(v) $G$ is nilpotent and $p$-abelian.

According to a previous result by Lee [*Group rings whose symmetric elements are Lie nilpotent.* Proc. Amer. Math. Soc. **127** (1999), 3153-3159] characterizing group algebras with classical involution whose symmetric elements are Lie nilpotent we have

**Corollary 2**

Let $KG$ be the group algebra of a torsion group $G$ over a field $K$ of characteristic $p \neq 2$ endowed with the classical involution. Then

$$\mathcal{U}^+(KG) \text{ is nilpotent } \iff KG^+ \text{ is Lie nilpotent.}$$

# Engel properties

**Theorem [Lee-Spinelli, 2010]**

Let $KG$ be the group algebra of a torsion group $G$ over an infinite field $K$ of characteristic $p > 2$ endowed with the classical involution. Then $\langle \mathcal{U}^+(KG) \rangle$ is $n$-Engel if, and only if, one of the following statements occurs:

(i) $Q_8 \not\subseteq G$, $G$ is nilpotent and it has a $p$-abelian normal subgroup of $p$-power index;

(ii) $Q_8 \subseteq G$ and $G \cong Q_8 \times E \times P$, where $E^2 = 1$ and $P$ is a $p$-group of bounded exponent having a $p$-abelian subgroup of finite index.

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## Corollary 1

Let $KG$ be the group algebra of a torsion group $G$ such that $Q_8 \not\subseteq G$ over an infinite field $K$ of characteristic $p \neq 2$ endowed with the classical involution. The following statements are equivalent:

(i) $\langle \mathcal{U}^+(KG) \rangle$ is bounded Engel;
(ii) $KG^+$ is bounded Lie Engel;
(iii) $KG$ is bounded Lie Engel;
(iv) $\mathcal{U}(KG)$ is bounded Engel;
(v) either
  (a) $p = 0$ and $G$ is abelian or
  (b) $G$ is nilpotent and it has a $p$-abelian normal subgroup of $p$-power index.

According to a previous result by Lee [*The Lie n-Engel property in group rings.* Comm. Algebra **28** (2000), 867-881] characterizing group algebras with classical involution whose symmetric elements are Lie $n$-Engel we have

## Corollary 2

Let $KG$ be the group algebra of a torsion group $G$ over an infinite field $K$ of characteristic $p \neq 2$ endowed with the classical involution. Then

$$\langle \mathcal{U}^+(KG) \rangle \text{ is } n\text{-Engel} \iff KG^+ \text{ is Lie } m\text{-Engel.}$$

# Solvability identities

The question was studied by Lee-Spinelli [*Group rings whose symmetric units are solvable.* Comm. Algebra **37** (2009), 1604-1618] and solved for arbitrary infinite fields $K$ and groups $G$ when

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- $P := \{x | \ x \in G \ \ x^{p^r} = 1\}$ is finite.

**Theorem [Lee-Spinelli, 2009]**

Let $KG$ be the group algebra of a group $G$ over an infinite field $K$ of characteristic $p \neq 2$. If $P$ is infinite and $G$ does not contain elements whose order divides $p^2 - 1$, then the following statements are equivalent:

(i) $\mathcal{U}^+(KG)$ is solvable;

(ii) $\mathcal{U}(KG)$ is solvable;

(iii) $G$ is $p$-abelian.

The question of when $KG^+$ is Lie solvable is more delicate and was studied in a paper by Lee-Sehgal-Spinelli [*Group algebras whose symmetric and skew elements are Lie solvable.* Forum Math. **21** (2009), 661-671]. According to this result we have

**Corollary [Lee-Sehgal-Spinelli, 2009]**

Let $KG$ be the group algebra of a group $G$ over an infinite field $K$ of characteristic $p \neq 2$ endowed with the classical involution. If $P$ contains an infinite subgroup of bounded exponent and $G$ does not contain elements whose order divides $p^2 - 1$, then the following statements are equivalent:

(i) $\mathcal{U}^+(KG)$ is solvable;
(ii) $\mathcal{U}(KG)$ is solvable;
(iii) $KG^+$ is Lie solvable;
(iv) $KG$ is Lie solvable;
(v) $G$ is $p$-abelian.

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## Theorem [Giambruno-Polcino Milies-Sehgal, 2009]

Let $KG$ be the group algebra of a torsion group $G$ over an infinite field $K$ of characteristic $p \neq 2$ endowed with a $K$-linear involution. Then $\mathcal{U}^+(KG)$ is GI if, and only if,

(a) $KG$ is semiprime and $G$ is either abelian or an $SLC$-group, or

(b) $KG$ is not semiprime, $P$ is a normal subgroup of $G$, $G$ has a $p$-abelian normal subgroup of finite index and either
- $G'$ is a $p$-group of bounded exponent or
- $G/P$ is an $SLC$-group and $G$ contains a normal $*$-invariant $p$-subgroup $B$ of bounded exponent such that $P/B$ is central in $G/P$ and the induced involution acts as the identity on $P/B$.

# $SLC$-groups

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A group $G$ is called an *LC-group* (that is, it has the *"lack of commutativity" property*) if it is not abelian, but, whenever $g, h \in G$ and $gh = hg$, then at least one of $\{g, h, gh\}$ is central.
A group $G$ is an $LC$-group with a unique nonidentity commutator if, and only if, $G/\zeta(G) \simeq C_2 \times C_2$.

**Definition**

A group $G$ endowed with an involution $\star$ is said to be a *special LC-group*, or *SLC-group*, if it is an $LC$-group, it has a unique nonidentity commutator $z$ and, for all $g \in G$, we have $g^\star = g$ if $g \in \zeta(G)$ and, otherwise, $g^\star = zg$.

**Theorem [Jespers-Ruiz Marin, 2006]**

Let $R$ be a commutative ring of characteristic different from 2, and $G$ a nonabelian group endowed with an involution $\star$. Then $RG^+$ is commutative if, and only if, $G$ is an SLC-group.

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- By [GPMS] we know that $P \trianglelefteq G$.
- Let $N \trianglelefteq G$ and $\star$-invariant. If $\mathcal{U}^+(KG)$ satisfies $w$, then $\mathcal{U}^+(K(G/N))$ satisfies $w$.

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- $\mathcal{U}^+(K(G/P))$ is nilpotent and $K(G/P)$ is semiprime.

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- Let $N \trianglelefteq G$ and $\star$-invariant. If $\mathcal{U}^+(KG)$ satisfies $w$, then $\mathcal{U}^+(K(G/N))$ satisfies $w$.
- $\mathcal{U}^+(K(G/P))$ is nilpotent and $K(G/P)$ is semiprime.
- By [GPMS] $G/P$ is abelian or $G/P$ is an $SLC$-group.

# $G$ finite

By [GPMS] $G$ is locally finite. Hence it is relevant to study the case in which $G$ is finite.

**Lemma**

Let $G$ be a finite group. If $\mathcal{U}^+(KG)$ is nilpotent, then $G$ is nilpotent and $G/P$ is either abelian or an $SLC$-group.

# $G/P$ abelian

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**Theorem [Lee-Sehgal-Spinelli, 2010]**

Let $G/P$ be abelian. If $\mathcal{U}^+(KG)$ is nilpotent, then $G$ is nilpotent and $p$-abelian (hence, $\mathcal{U}(KG)$ is nilpotent).

# $G/P$ is an $SLC$-group

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**Theorem [Lee-Sehgal-Spinelli, 2010]**

Let $G/P$ be an $SLC$-group. Then $\mathcal{U}^+(KG)$ is nilpotent if, and only if, $G$ is nilpotent and $G$ has a finite normal $*$-invariant $p$-subgroup $N$ such that $G/N$ is an $SLC$-group.

# Main Theorem

**Theorem [Lee-Sehgal-Spinelli, 2010]**

Let $K$ be an infinite field of characteristic $p > 2$ and $G$ a torsion group having an involution $*$, and let $KG$ have the induced involution. Suppose that $\mathcal{U}(KG)$ is not nilpotent. Then $\mathcal{U}^+(KG)$ is nilpotent if, and only if, $G$ is nilpotent and $G$ has a finite normal $*$-invariant $p$-subgroup $N$ such that $G/N$ is an $SLC$-group.

According to the result by Passi-Passman-Sehgal (1973) and Khripta (1972) $KG$ is Lie nilpotent if, and only if, $\mathcal{U}(KG)$ is nilpotent.

By using the results of Lee-Sehgal-Spinelli [*Lie properties of symmetric elements in group rings II.* J. Pure Appl. Algebra **213** (2009), 1173-1178], one has

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