On the adjoint groups of radical rings and related questions

Yaroslav P. Sysak (Kiew)

Ukrainian National Academy of Sciences

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Let $R = (R, +, \cdot)$ be an associative ring, not necessarily with an identity element. Then the set of all elements of R forms a monoid with neutral element $0 \in R$ under the "circle" operation $r \circ s = r + s + rs$ for all $r, s \in R$ which is called the *adjoint monoid* of R. The group of all invertible elements of this monoid is said to be the *adjoint group* of R and usually denoted by R°. The ring R is called *radical* (in sense of N. Jacobson) if $R = R^\circ$ which means that R coincides with its Jacobson radical.

If R is a ring with an identity 1, then the set of all non-zero elements of R forms the *multiplicative monoid* of R whose group of all invertible elements is called the *multiplicative group* of R and denoted by R^*. In this case the mapping $r \mapsto 1+r$ with $r \in R^\circ$ determines a group isomorphism from R° onto R^*. In fact, $R^* = 1+R^\circ$. Recall that R is a *division ring* if $R^* = R \setminus \{0\}$. It is well-known that every ring R can be embedded in a ring R_1 with 1, for instance, if we put $R_1 = R \oplus \mathbb{Z}$ and extend the multiplication in R_1 by distributivity.

The best known examples of radical rings are the nilpotent rings. A ring R is *nilpotent* if there exists a positive integer n such that every product of $n+1$ elements from R is 0. For instance, rings of strictly upper triangular square matrices over a ring are well-known examples of nilpotent rings. If every finitely generated subring of R is nilpotent, then R is *locally nilpotent*. It is obvious that the adjoint group of every (locally) nilpotent ring is (locally) nilpotent.

A ring R is *nil*, if every element r of R is nilpotent, i.e. there exists a positive integer $n = n(r)$ with $r^n = 0$. Every locally nilpotent ring is nil. In 1964, for any prime p, E.S. Golod constructed a finitely generated nil algebra over the Galois field $GF(p)$ which is not nilpotent and whose adjoint group is a p-group containing finitely generated infinite periodic subgroups.

Triply factorized groups associated with radical rings

Let P be a right ideal of the radical ring R, so that $M = R/P$ is a right R-module. Then $A = R^\circ$ operates on M via the rule $m^a = m + ma$ for a in A and m in M. If $G = M \rtimes A$ is the semidirect product of M by A and $B = \{am \mid m = a + P,\ a \in A\}$, then B is a subgroup of G and

$$G = M \rtimes A = M \rtimes B = AB$$

with $A \cap B$ isomorphic to P°.

If in particular $P = 0$, then M is isomorphic to R^+ and $A \cap B = 1$. In this case G is isomorphic to the matrix group

$$\begin{pmatrix} 1 & R \\ 0 & 1 + R \end{pmatrix}$$

in which $M = \begin{pmatrix} 1 & R \\ 0 & 1 \end{pmatrix}$, $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 + R \end{pmatrix}$ and $B = \{\begin{pmatrix} 1 & r \\ 0 & 1 + r \end{pmatrix} \mid r \in R\}$.

The adjoint group of the matrix ring $M_2(R)$ is isomorphic to

$$G(R) = \begin{pmatrix} 1+R & R \\ R & 1+R \end{pmatrix}$$

and the following assertion can directly be verified.

Let $t_{ij}(r)$ be the transvection with the element r in the position (i,j).

Theorem 1. *If* $A = \begin{pmatrix} 1+R & 0 \\ 0 & 1+R \end{pmatrix}$, $B = t_{12}(-1)At_{12}(1)$ *and* $C = t_{21}(-1)At_{21}(1)$, *then* $G(R) = ABC$ *and* $AB = BA$, $AC = CA$ *and* $BC = CB$.

Clearly if R is commutative, then the subgroups A, B and C of $G(R)$ are abelian.

Corollary 2. *Let* $\mathbb{Q}_p$ *be the set of all rational numbers whose denominators are not divisible by* p. *Then* $R = p\mathbb{Q}_p$ *is a radical subring of* $\mathbb{Q}$ *and the group* $G(R)$ *contains a non-abelian free subgroup generated by matrices* $\begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix}$ *and* $\begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix}$. *Therefore* $G(R)$ *is a non-soluble linear group over* $\mathbb{Q}$ *which is a product of three pairwise permutable abelian subgroups.*

Corollary 3. *Let* p *be a prime and* $R = p\mathbb{Z}_{p^m}$ *for* p *odd or* $R = 4\mathbb{Z}_{2^m}$ *for* $p = 2$ *where* $m \geq 3$. *Then the factor group* $G(R)/Z(G(R))$ *is a product of three pairwise permutable cyclic subgroups of orders* p^{m-1} *or* 2^{m-2}, *respectively, and its derived length is at least* $log_2(m)$.

The structure of the adjoint group of a radical ring

The next theorem proved by B. Amberg, O. Dickenschied and myself (1998) shows that the adjoint group of any radical ring satisfies some solubility condition. Recall that a group G is an SN-*group* if it has a series with abelian factors which is equivalent to the property that every finitely generated subgroup of G is different from its derived subgroup.

In what follows, if the other is not claimed, R is always a radical ring.

Theorem 4. *The adjoint group R° of R is an SN-group in which every finite subgroup is nilpotent.*

Using Zelmanov's theorem on the Restricted Burnside Problem, from Theorem 6 one can deduce the following.

Corollary 5. *Let G be a subgroup of R° and suppose that one of the following conditions holds:*

- *G is locally finite,*
- *G has finite exponent,*
- *G is an n-Engel group for some $n \geq 1$,*
- *G is locally artinian.*

Then the group G is locally nilpotent.

It should be noted that an essential contribution to the theory of Engel groups was made by K. Gruenberg.

The question of B.I. Plotkin in Kourovka Notebook (1969, Question 2.65) whether R° has a series with central factors was negatively answered by O.M. Neroslavskii (1973). His counterexample was constructed in the following way.

Let F be the algebra of formal power series without constant terms in two non-commuting indeterminates x and y over $\mathbb{F}_p$ with $p > 2$. Then the elements $u = (1+x)^{-1}y(1+x) - y^2 - 2y$ and y^p of F generate the ideal I in F such that the factor algebra $R = F/I$ is a radical algebra whose adjoint group R° contains a subgroup which is isomorphic to the group $G = < b > \rtimes < a >$ with $a^{-1}ba = b^2$ where a is of infinite order and b has order p. In particular G is metacyclic, but not nilpotent.

As Golod's example shows, the periodic subgroups of R° need not be locally nilpotent. However the following assertion holds.

Theorem 6. *If G is a periodic subgroup of R° and Z is the center of G, then the factor group G/Z is the direct product of its primary components. In particular, every two elements of coprime orders of G are permutable.*

The question whether every periodic subgroup of R° is a direct product of its primary components is open. It can be reduced to the following more general problem which is of independent interest.

Does every central extension G of a cyclic p-group by a periodic p'-group split if G is an SN-group?

It should be noted that if G is not an SN-group, then an example of S.I. Adjan (The Burnside problem and identities in groups, Springer-Verlag, Berlin (1978)., p. 276, VII.1.9) shows that there exists such a non-split extension.

The following result was obtained by P. Cohn (1971 - 1973). We reformulate his result as follows.

Theorem 7. *Every radical subalgebra of a division ring can be embedded in a radical algebra R (again contained in a division ring) whose adjoint group R° has only two conjugacy classes of elements.*

This means in particular that there exist SN-groups in which every two non-identity elements are conjugate.

Recall that a group is *hyperabelian* if it has an ascending normal series with abelian factors. The following result was obtained by F. Catino, M.M. Miccoli and myself (2007).

Theorem 8. *If R has no non-zero nilpotent ideal and Z is the center of R, then the factor group R°/Z° has no non-trivial soluble-by-finite normal subgroup.*

As a corollary, we have

Corollary 9. *Let H be a hyperabelian normal subgroup of R°. Then the derived subgroup H' is locally nilpotent.*

An abelian group G is said to be of *finite torsion-free rank* if it has a finitely generated torsion-free subgroup A such that the factor group G/A is periodic.

Corollary 10. *If the adjoint group R° is hyperabelian and the commutator factor group of R° has finite torsion-free rank, then the ring R (and so the adjoint group R°) is locally nilpotent. In particular, every radical ring whose adjoint group is finitely generated and hyperabelian must be nilpotent.*

In this connection a natural question arises

whether every radical ring R with finitely generated adjoint group is nilpotent.

In the case when R° is generated by two elements, the answer is positive and essentially depends on the following result of Amberg and L.S. Kazarin (2003).

Lemma 11. *Let G be a finite p-group with two generators. If G occur as the adjoint group of a nilpotent p-algebra, then the order of G is bounded by p^5.*

Nilpotent groups isomorphic to the adjoint group of a radical ring

Clearly every abelian group A occurs as the adjoint group of the ring with trivial multiplication on A.

L. Kaloujnine (1954) has shown that every finite p-group of class 2 with odd p is isomorphic to the adjoint group of some nilpotent ring. In fact, all groups of order p, p^2 and p^3 occur as the adjoint group of some nilpotent ring, but a group of order p^4, if and only if, it is nilpotent of class ≤ 2 (R.L. Kruse and D.T. Price, Nilpotent rings, 1967, Chapter I, Section 6). Groups of class 3 and of order p^5 for odd primes p are described by K.I. Tahara and A. Hosomi (1983).

Every finitely generated nilpotent group of class at most 2 is the adjoint group of some nilpotent ring (R. Sandling, 1974).

A.W. Hales and I.B.S. Passi (1978) have shown that a nilpotent group G of class 2 is the adjoint group of a nilpotent ring R with $R^2 = 0$ provided that its commutator factor group G/G' is either a direct sum of cyclic groups, or divisible, or torsion, or torsion-free and completely decomposable. Furthermore, they constructed a torsion-free nilpotent group G of class 2 with Prüfer rank $r(G) = 3$ which cannot be the adjoint group of such a nilpotent ring and asked "whether this G occurs as the adjoint group of some nilpotent ring R with $R^n = 0$ for $n > 2$ (or, for that matter, of any radical ring)". The answer is negative because radical ring R whose adjoint group R° is torsion-free of rank n is nilpotent with $R^n = 0$ (Amberg and myself, 2001).

Adjoint groups of finite nilpotent p-algebras

Associative algebras over the Galois field $GF(p)$ for some prime p are called p-*algebras*. The adjoint group of every nilpotent p-algebra is a p-group.

Which finite p-groups can occur as the adjoint group R° of a nilpotent p-algebra R?

Clearly R° is elementary abelian p-group if $R^2 = 0$.

If R° is cyclic, then either $|R^\circ| = p$ or $|R^\circ| = 4$ (I. Fischer and K. Eldridge, 1969).

If R° is metacyclic, then R° is either elementary abelian of order at most p^2, or $p = 3$ and $R^\circ \simeq \mathbb{Z}_9 \times \mathbb{Z}_3$, or $p = 2$ and R° is one of the following: $\mathbb{Z}_4, \mathbb{Z}_2 \times \mathbb{Z}_4, \mathbb{Z}_4 \times \mathbb{Z}_4, \mathbb{Z}_2 \times \mathbb{Z}_8, \mathbb{Z}_4 \rtimes \mathbb{Z}_4, D_8, Q_8$ (B.O. Gorlov, 1995).

If R° is 2-generated and $p > 2$, then either R° is a metacyclic group or a non-abelian group of order p^3 and of exponent p; if $p = 2$, then $|R^\circ| \le 2^5$ (Amberg, Kazarin, 1999).

The case of a commutative nilpotent p-algebra is of special interest. A conjecture of N.H. Eggert (1971) says that for every commutative nilpotent p-algebra R we have
$\dim R \ge p \dim R^{(1)}$, where $R^{(1)} = \{x^p \mid x \in R\}$.
In some special cases it was affirmed by R. Bautista (1976), C. Stack (1996) and some others authors. A proof of L. Hammoudi (Pacific J. Math. 202 (2002), 93 -97) has an error which was admitted by himself. The Eggert conjecture can also be reformulated as follows:

if R is a commutative p-algebra, then $r(R^+) \le \frac{p}{p-1} r(R^\circ)$.

This implies in particular that if $G = AB$ is a finite p-group with abelian subgroups A and B, then $r(G) \le 2(r(A) + r(B))$. At the present time it is even unknown whether there exists a linear function f such that $r(G) \le f(r(A), r(B))$.

Relations between the adjoint and Lie structures in radical rings

Every ring R can be viewed as a Lie ring under the Lie multiplication $[r,s] = rs - sr$ for all $r,s \in R$ which is called the *associated Lie ring* of R and denoted by $R^{(-)}$.

General question. *Which relations exist between the group structure of R° and the Lie structure of R and what influence do they have on the ring structure of R?*

An obvious example of such relations is the following:

R^{ad} is abelian if and only if the Lie ring $R^{(-)}$ is abelian.

If $r_1, r_2, \ldots$ are elements of R, the Lie-commutators $[r_1, \ldots, r_{n+1}]$ are defined inductively by $[r_1, \ldots, r_{n+1}] = [[r_1, \ldots, r_n], r_{n+1}]$ for all $n \geq 2$.

The ring R is called *Lie-nilpotent* if $R^{(-)}$ is nilpotent, i.e. there exists a positive integer n such that

$[r_1, \ldots, r_{n+1}] = 0$ for all $r_1, \ldots r_{n+1}$ of R.

The least n with this property is the *class of Lie-nilpotency* of R. We will also say that R is *locally Lie-nilpotent* if every finitely generated subring of R is Lie-nilpotent. Obviously in this case the Lie ring $R^{(-)}$ is locally nilpotent.

It was proved by S.A. Jennings (1955) that a radical ring R is Lie-nilpotent if and only if its adjoint group R° is nilpotent. He conjectured also that the classes of nilpotency of both structures coincide. This was confirmed by X. Du (1992) even in more strong form, namely

$Z_n(R^{(-)}) = Z_n(R^{\circ})$ *for each positive integer* n.

It is trivial for $n = 1$ and was shown by H. Laue (1984) for $n = 2$ before. Later Du (2001) proved also that if $2R = R$, then $Z_\alpha(R^{(-)}) = Z_\alpha(R^{\circ})$ for every ordinal number α. It is unknown at present whether the condition $2R = R$ can be removed.

A natural generalization of the concept of nilpotency of groups is that of nilpotency of semigroups in the sense of A. Mal'cev or B.H. Neumann and T. Taylor whose precise definition can be given as follows.

Let X be a countable set of non-commuting indeterminates and let the sequence $W_1(x, y)$, $W_2(x, y, z_1)$,..., $W_n(x, y, z_1,$ ... of words in the indeterminates x, y, $z_1, \ldots,$ $z_n, \ldots$ of X be defined by the rule

$W_1(x, y) = xy$ and $W_{n+1}(x, y, z_1, \ldots, z_n) = W_n(x, y, z_1, \ldots, z_{n-1})z$

for every $n \geq 1$. A *semigroup* A is said to be *nilpotent* if there exists a positive integer n such that

$$W_n(a, b, c_1, \ldots, c_{n-1}) = W_n(b, a, c_1, \ldots, c_{n-1})$$

for any elements $a, b, c_1, \ldots, c_{n-1}$ of A. The least n with this property is called the *class of nilpotency* of the semigroup A.

In answer to a question posed by A.N. Krasil'nikov (1997) and independently by D. Riley and V. Tasić (1999) it was proved by Amberg and myself (2001) that

for each positive integer n, *the adjoint monoid of a ring* R *is nilpotent of class* n *if and only if* R *is Lie-nilpotent of class* n.

Furthemore, we also proved (2003) that

the adjoint monoid of R *is locally nilpotent if and only if* R *is locally Lie-nilpotent.*

The ring R is *Engel* if $[r, s, \ldots, s] = 0$ for each pair of elements r and s in R, and n-*Engel* if s appears exactly n times.

Theorem 12. *(Amberg and myself, 2000) The adjoint group R° of R is an n-Engel group for some positive integer n if and only if R is an m-Engel ring for some positive integer m depending only on n.*

It is trivial that $n = 1$ implies $m = 1$. The question which relationship exists between n and m was considered by Dickenshied in his dissertation (1997). He proved that $m = n$ if $n = 2$ or $n = 3$ and R^+ contains no elements of order 2. No other relations are known.

Since

$$[x, {}_n y] = \sum_{i=0}^{n} (-1)^{i+1} \binom{n}{i} y^i x y^{n-i}.$$

for all $x, y \in R$ and $n \geq 1$, every nil ring is Engel. This leads to the question whether the adjoint group of every nil ring is Engel.

For any additive subgroups V and W of R, let $[V, W]$ be the additive subgroup of R generated by all Lie-commutators $[v, w]$ with $v \in V$ and $w \in W$. The derived chain of a Lie ring R is defined inductively as

$\delta_0(R) = R$ and $\delta_{n+1}(R) = [\delta_n(R), \delta_n(R)]$ for each integer $n \geq 0$.

The ring R is called *Lie-soluble of length at most* m if $\delta_m(R) = 0$. Lie-soluble rings of length at most 2 are called *Lie-metabelian*.

A.E. Zalesskii and M.B. Smirnov (1982) and independently R.K. Sharma and J.B. Srivastava (1985) proved that every Lie-soluble ring R has a nilpotent ideal I whose factor ring R/I is center-by-metabelian as a Lie ring. However the adjoint group R° of R need not be soluble in this case.

Indeed, if R is the ring of all 2×2-matrices over any commutative radical domain S of characteristic 2, say $S = \{\frac{xf(x)}{1+xf(x)} \mid f(x) \in \mathbb{F}_2[x]\}$, then R is center-by-metabelian and R° contains free subgroup generated by the matrices $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$.

On the other hand, Krasil'nikov (1992) and independently Sharma and Srivastava (1992) proved that the adjoint group of every Lie-metabelian ring is metabelian.

By analogy with the nilpotent case, the question arises whether every radical ring with soluble adjoint group is Lie-soluble.

A positive answer for nil algebras over infinite fields was obtained by Smirnov (1985). Furthermore, Krasil'nikov (1992) proved that every nil ring with metabelian adjoint group is Lie-metabelian.

The following theorem of Amberg and myself (2002) yields a complete answer to this question.

Theorem 13. *Let R be a radical ring. Then the adjoint group R° is soluble if and only if the following statements hold:*

(1) R *is Lie-soluble, and*

(2) there exists a chain

$$0 = I_0 \subseteq I_1 \subseteq \ldots \subseteq I_m = R$$

of ideals of R such that every factor I_i/I_{i-1} is generated by commutative ideals of R/I_{i-1} for $1 \leq i \leq m$.

Moreover, if R° is soluble of length n for some $n \geq 1$ and L the Levitzki radical of R, then there exist positive integers k, l and m depending only on n such that:

(3) R *satisfies the identity $[x, y]^k = 0$ for all $x, y \in R$;*

(4) the factor ring R/L is commutative and L satisfies the identity $[x, y, \ldots, y] = 0$ with y repeated l times, i.e. L is an l-Engel ring;

(5) the derived subgroup of R° is an m-Engel group.

Furthermore, in answer to a question of Krasilnikov and Sharma and Srivastava (1992), Amberg and myself (2004) proved the following.

Theorem 14. *The adjoint group R° of a radical ring R is metabelian if and only if R is Lie metabelian.*

Finally, the ring R is *Lie-supersoluble* if R has an ascending series of Lie-ideals whose factors are cyclic as additive groups. The next assertion proved by Catino, Miccoli and myself (2009)

Theorem 15. *If R is a semilocal ring (i.e. R is artinian modulo its Jacobson radical) whose adjoint group is locally supersoluble, then R is locally Lie-supersoluble and contains a locally Lie-nilpotent ideal I of finite index such that the factor ring R/I is a direct sum of ideals each of which is isomorphic either to the Galois field $\mathbb{F}_p$ of prime order p or the matrix algebra $M_2(\mathbb{F}_2)$.*

In particular, if R is radical, then R° is locally supersoluble if and only if R° is locally nilpotent.