On complete resolutions

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A $\mathbb{Z}G$-module M is said to admit a complete resolution $(\mathcal{F}, \mathcal{P}, n)$ if there is an acyclic complex $\mathcal{F} = \{(F_i, \vartheta_i)|\ i \in \mathbb{Z}\}$ of projective modules, and a projective resolution $\mathcal{P} = \{(P_i, d_i)|\ i \in \mathbb{Z}, i \geq 0\}$ of M such that $\mathcal{F}$ and $\mathcal{P}$ coincide in dimensions greater than n

$$\begin{array}{l} \mathcal{F}\colon \cdots \to F_{n+1} \to F_n \overset{\vartheta_n}{\to} F_{n-1} \to \cdots \to F_0 \to F_{-1} \to F_{-2} \to \cdots \\ \qquad\qquad\ \ \| \qquad\quad\ \ \| \\ \mathcal{P}\colon \cdots \to P_{n+1} \to P_n \overset{d_n}{\to} P_{n-1} \to \cdots \to P_0 \to M \to 0 \end{array}$$

The number n is called the coincidence index of the complete resolution.

Ikenaga (1984) defined generalized Tate cohomology for the class of groups G which admit

- complete resolutions

and for which the generalized cohomological dimension of G

- $\underline{\text{cd}}G < \infty$

where

$\underline{\text{cd}}G =\sup\{k\text{: } \text{Ext}^k_{\mathbb{Z}G}(A,F)\neq 0,\ A\ \mathbb{Z}-\text{free},\ F\ \mathbb{Z}G-\text{free}\}$

$\widehat{H}^i(G,B) = H^i(\text{Hom}_{\mathbb{Z}G}(\mathcal{F},B)), i \in \mathbb{Z}$,
where $\mathcal{F}$ is a complete resolution of the trivial $\mathbb{Z}G$-module $\mathbb{Z}$
generalized Tate cohomology

$\underline{\text{cd}}G \leq n$ is equivalent to the following extension condition

For every exact sequence of $\mathbb{Z}G$-modules

$$\begin{array}{cccccccccccc} 0 \to & \ker d_n & \to & P_n & \xrightarrow{d_n} & P_{n-1} & \to \cdots \to & P_0 & \to & A & \to & 0 \\ & \downarrow & \swarrow & & & & & & & & & \\ & P & & & & & & & & & & \end{array}$$

P_i projective $0 \leq i \leq n$ A $\mathbb{Z}$-free, any map $\ker d_n \to P$, P projective, extends to a map $P_n \to P$.

If G admits a complete resolution and $\underline{\text{cd}}G < \infty$ then $\underline{\text{cd}}G = \min\{n|\ n$ coincidence index of a complete resolution of $G\}$

The generalized Tate cohomology defined by Ikenaga coincides with

- Tate cohomology for finite groups

- Farrell-Tate cohomology for groups of finite virtual cohomological dimension, vcd$G < \infty$

- If G is a virtually torsion free group and $\text{vcd}G < \infty$ then $\text{vcd}G = \underline{\text{cd}}G$

- $\underline{\text{cd}}G = \sup\{k\text{: } \text{Ext}^k_{\mathbb{Z}G}(A,F) \neq 0,\ A\ \mathbb{Z}\text{–free},\ F\ \mathbb{Z}G\text{–free}\}$

$$\underline{\text{cd}}G \leq \text{silp}\,\mathbb{Z}G \leq \underline{\text{cd}}G + 1$$

Gedrich and Gruenberg (1987) in their study of complete cohomological functors considered the invariants $\text{silp}\,\mathbb{Z}G$ and $\text{spli}\,\mathbb{Z}G$ and showed

- $\text{silp}\,\mathbb{Z}G \leq \text{spli}\,\mathbb{Z}G$ and if $\text{spli}\,\mathbb{Z}G < \infty$ then $\text{silp}\,\mathbb{Z}G = \text{spli}\,\mathbb{Z}G$

- If $\text{spli}\,\mathbb{Z}G < \infty$ then G admits a complete resolution
  so generalized Tate cohomology or complete cohomology is defined for G.

Mislin (1994) defined complete cohomology or generalized Tate cohomology <u>for any group G via satellites</u> as follows

$$\widehat{\mathrm{Ext}}^{i}_{\mathbb{Z}G}(A,B) = \quad \lim_{\to j\geq 0} S^{-j}\mathrm{Ext}^{i+j}_{\mathbb{Z}G}(A,B)$$

where $S^{-j}\mathrm{Ext}^{i+j}_{\mathbb{Z}G}(A,\_)$ denotes the j-th left satellite of the functor $\mathrm{Ext}^{i+j}_{\mathbb{Z}G}(A,\_)$.

The family $\{\hat{H}^i(G,\_):\ i\in\mathbb{Z}\}$ forms a cohomological functor which is the P-completion of ordinary cohomology (or completion with respect to projective modules)

i.e. $\hat{H}^i(G,\text{projective})=0$ for all $i\in\mathbb{Z}$ and there is a morphism $H^i(G,\_)\xrightarrow{\tau}\hat{H}^i(G,\_)$ such that if $\{V^i,\ i\in\mathbb{Z}\}$ is a cohomological functor with $V^i(\text{projective})=0$ for all $i\in\mathbb{Z}$, and $H^i(G,\_)\xrightarrow{\sigma} V^i$ a morphism of cohomological functors, then σ factors uniquely through τ.

Alternative but equivalent definitions were also introduced by Benson and Carlson (1992) and Vogel (1992)

$\mathrm{pd}_{\mathbb{Z}G} M < \infty$ if and only if $\widehat{\mathrm{Ext}}^0_{\mathbb{Z}G}(M, M) = 0$ Kropholler (1993).

It was shown by Cornick and Kropholler (1998) that if a group G admits a complete resolution in the strong sense $(\mathcal{F}, \mathcal{P}, n)$, i.e. $\mathrm{Hom}_{\mathbb{Z}G}(\mathcal{F}, P)$ is acyclic for every projective $\mathbb{Z}G$-module P then complete cohomology can be calculated using complete resolutions in the strong sense, i.e.

$$\widehat{H}^i(G, B) \simeq H^i(\mathrm{Hom}_{\mathbb{Z}G}(\mathcal{F}, B)) \ i \in \mathbb{Z}$$

The advantage with this approach is that one has computational devices such as
Eckmann - Shapiro lemma
certain spectral sequences

Complete cohomology is not always calculated via complete resolutions as they do not always exist

If G admits a complete resolution of coincidence index n then
findim $\mathbb{Z}G \leq n$ (Mislin + T, 2000)
The finitistic dimension of $\mathbb{Z}G$, findim $\mathbb{Z}G$, is the supremum of the projective dimensions of the $\mathbb{Z}G$-modules of finite projective dimension

Any group which contains a free abelian subgroup of infinite rank does not admit a complete resolution

For any group G

$$\operatorname{findim} \mathbb{Z}G \leq \operatorname{silp} \mathbb{Z}G \leq \operatorname{spli} \mathbb{Z}G$$

Emmanouil (2008) $\operatorname{spli} \mathbb{Z}G \leq \operatorname{silp} \mathbb{Z}G$

The kernels in a complete resolution $(\mathcal{F}, \mathcal{P}, n)$ in the strong sense are the Gorenstein projective modules over $\mathbb{Z}G$.

The Gorenstein projective dimension of a $\mathbb{Z}G$-module M, $\operatorname{Gpd}_{\mathbb{Z}G} M$, is defined via resolutions by Gorenstein projective modules, i.e.
$\operatorname{Gpd}_{\mathbb{Z}G} M \leq n$ if and only if M has a Gorenstein projective resolution of length n.

The Gorenstein projective dimension was defined by Enochs and Jenda (1995) and is related to the G-dimension defined by Auslander (1966) for finitely generated modules over commutative Noetherian rings.

For any group G
$\mathrm{Gcd}_{\mathbb{Z}}G = \underline{\mathrm{cd}}G$ (Bahlekeh, Dembegioti, T 2009)

$$\underline{\mathrm{cd}}G \leq \mathrm{silp}\,\mathbb{Z}G = \mathrm{spli}\,\mathbb{Z}G \leq \underline{\mathrm{cd}}G + 1$$

If $\mathrm{Gcd}_{\mathbb{Z}}G < \infty$ then
$\mathrm{Gcd}_{\mathbb{Z}}G = \sup\{i : \ H^i(G, P) \neq 0, \ P \text{ projective}\}$
(Holm, 2004)

Thm: Let G be an $\mathbf{H}\mathfrak{F}$-group of type FP_∞. Then there is a finite dimensional model for $\underline{E}G$, the classifying space for proper actions (Kropholler - Mislin, 1993).

Def: A group G is said to be of type Φ if it has the property that for every $\mathbb{Z}G$-module M, $\mathrm{pd}_{\mathbb{Z}G}M < \infty$ if and only if $\mathrm{pd}_{\mathbb{Z}H}M < \infty$ for every finite subgroup H of G (T, 2007)

Thm: If G is a group such that $N_G(H)/H$ is of type Φ for every finite subgroup H of G and $\dim|\Lambda(G)| < \infty$, then G admits a finite dimensional model for $\underline{E}G$.
$|\Lambda(G)|$ is the G-simplicial complex determined by the poset of the non trivial finite subgroups of G.

Properties of $\mathrm{Gcd}_{\mathbb{Z}}G$ (Bahlekeh, Dembegioti, T, 2009)

(1) If $H \leq G$ then $\mathrm{Gcd}_{\mathbb{Z}}H \leq \mathrm{Gcd}_{\mathbb{Z}}G$.

(2) If $1 \to N \to G \to K \to 1$ is an extension of groups, then $\mathrm{Gcd}_{\mathbb{Z}}G \leq \mathrm{Gcd}_{\mathbb{Z}}N + \mathrm{Gcd}_{\mathbb{Z}}K$.

(3) If $1 \to N \to G \to K \to 1$ is an extension of groups with $|N| < \infty$ then $\mathrm{Gcd}_{\mathbb{Z}}G = \mathrm{Gcd}_{\mathbb{Z}}K$.

(4) If $1 \to N \to G \to K \to 1$ is an extension of groups with $|K| < \infty$ then $\mathrm{Gcd}_{\mathbb{Z}}G = \mathrm{Gcd}_{\mathbb{Z}}N$.

(5) If F is a finite subgroup of G, and $N_G(F)$ its normalizer in G, then $\mathrm{Gcd}_{\mathbb{Z}}(N_G(F)/F) \leq \mathrm{Gcd}_{\mathbb{Z}}G$.

Conjecture A (T 2007)
The following are equivalent for a group G

(1) G admits a finite dimensional model for $\underline{E}G$.

(2) There is a resolution of finite length by permutation modules induced from finite subgroups of G

$$0 \to \bigoplus_{i_n \in I_n} \mathbb{Z}(G/G_{i_n}) \to \cdots \to \bigoplus_{i_n \in I_0} \mathbb{Z}(G/G_{i_0}) \to \mathbb{Z} \to 0$$

(3) G is of type Φ.

(4) $\mathrm{Gcd}_{\mathbb{Z}} G < \infty$

($\mathrm{Gcd}_{\mathbb{Z}} G = \underline{\mathrm{cd}} G$
$\underline{\mathrm{cd}} G \leq \mathrm{silp}\, \mathbb{Z}G = \mathrm{spli}\, \mathbb{Z}G \leq \underline{\mathrm{cd}} G + 1$)

(5) $\mathrm{findim}\, \mathbb{Z}G < \infty$

(1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5)
Kropholler Mislin (5) $\Rightarrow$ (1) if G is an $\mathbf{H}\mathfrak{F}$-group with dim$|\Lambda(G)| < \infty$.

The class $\mathbf{H}\mathfrak{F}$ was defined by Kropholler (1993) as the smallest class of groups which contains the class of finite groups and whenever a group G admits a finite dimensional contractible G-CW-complex with stabilizers in $\mathbf{H}\mathfrak{F}$, then G is in $\mathbf{H}\mathfrak{F}$.

Another characterization of the finiteness of $\mathrm{Gcd}_{\mathbb{Z}}G$
The following are equivalent for a group G

1. $\mathrm{Gcd}_{\mathbb{Z}}G < \infty$

2. G admits a complete resolution and every complete resolution of G is a complete resolution in the strong sense

3. Complete cohomology can be calculated using complete resolutions

4. The Eckmann-Shapiro lemma is valid for complete cohomology (Dembegioti, T to appear)

Conj. C
A group G admits a complete resolution if and only if G admits a complete resolution in the strong sense

Remark I

Part of Conj. A

$\text{Gcd}_{\mathbb{Z}} G < \infty$ if and only if

$$0 \to \bigoplus_{i_n \in I_n} \mathbb{Z}(G/G_{i_n}) \to \cdots \to \bigoplus_{i_n \in I_0} \mathbb{Z}(G/G_{i_0}) \to \mathbb{Z} \to 0$$

G_{i_j} finite

If $\text{vcd} G < \infty$ then $\text{vcd} G = \text{Gcd}_{\mathbb{Z}} G$

There are groups G such that

$1 \to F \to G \to K \to 1$

with $|F| < \infty$ $\text{cd}_{\mathbb{Z}} K < \infty$ and G does not have a torsion free subgroup of finite index. An example of such a group is the following

$$G = A *_{H,\varphi} B$$

where $A \simeq B \simeq (\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}_p) \lhd \mathbb{Z}$ and

$H \simeq \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}_p$.

$$A=\langle a_1,a_2,a_3,a,d/$$

$$[a_i,a_j]=[a_i,d]=[a,d]=d^p=1,a_1^a=a_2,a_2^a=a_3,a_3^a=a_1a_2^{-3}a_3^2\rangle$$

$$B=\langle b_1,b_2,b_3,b,e/$$

$$[b_i,b_j]=[b_i,e]=[b,e]=e^p=1,b_1^b=b_2,b_2^b=b_3,b_3^b=b_1b_2^{-3}b_3^2\rangle$$

$H = \langle a_1, a_2^p, a_3, d\rangle \leq A$ and $\varphi : H\to B$ with $\varphi(a_1) = b_1^p e,\ \varphi(a_2^p) = b_2,\ \varphi(a_3) = b_3^p,\ \varphi(d) = e$. Note that, and vcd$A =$ vcd$B = 4$.

It follows that $\langle d\rangle \leq \cap\{N \leq G|\ |G : N| < \infty\}$, hence G does not have a torsion free subgroup of finite index. Moreover, there is a group extension $1\to\langle d\rangle\to G\to K\to 1$ with cd$K < \infty$.

This group was constructed by Dyer (1968) as a counterexample to a conjecture related to residual finiteness.

A group G is finite if and only if spli $\mathbb{Z}G$=1 if and only if $\mathrm{Gcd}_{\mathbb{Z}}G = 0$. (Dembegioti, T 2008)

Conjecture B: spli $\mathbb{Z}G = \underline{\mathrm{cd}}G + 1$

(Dembegioti, T 2008)

Thm: $\mathrm{Gcd}_{\mathbb{Z}}G \leq 1$ if and only if the group G acts on a tree with finite stabilizers. (Bahlekeh, Dembegioti, T 2009)