Uniform $(2, k)$-generation of matrix groups of small rank

M. Chiara Tamburini

Dedicated to Karl Gruenberg

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Theorem

Suppose $G = \langle x_1, \dots, x_m \rangle$ *with* $\prod_{i=1}^m x_i = 1$.
Then:

$$\sum_{i=1}^{n} v(x_i) \geq v(G) + v(G^*).$$

Consider a triple

$$(x_1, x_2, x_3) \tag{1}$$

of elements $x_i \in \mathrm{GL}_n(\mathbb{F})$, with $x_1 x_2 x_3 = 1$, and assume that

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Scott's formula, applied to the conjugation action of G on $V = \mathrm{Mat}_n(\mathbb{F})$, gives:

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If equality holds, then the triple is said to be *linearly rigid*.

Consider another triple

$$(x_1', x_2', x_3') \quad \text{with} \quad \prod_{i=1}^{3} x_i' = 1$$

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If (x_1, x_2, x_3) is rigid,there exists $g \in \mathrm{GL}_n(\mathbb{F})$ which does the conjugations simultaneously (**Strambach and Völklein** (1999)).

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In particular

$$\langle x'_1, x'_2 \rangle = \langle x_1^g, x_2^g \rangle = \langle x_1, x_2 \rangle^g .$$

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Then $\langle x, y\rangle$ fixes a non-degenerate bilinear form J.
If $\sigma = id$, J is symmetric or skew-symmetric.
If σ has order 2, J is hermitian.

Hint of the proof.

Let J conjugate x^{σ} to $\left(x^{-1}\right)^{t}$, and y^{σ} to $\left(y^{-1}\right)^{t}$.

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It follows:

$$x^t J x^\sigma = J, \quad y^t J y^\sigma = J.$$

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A group is said to be $(2, k)$-generated if it can be generated by a pair of elements of respective orders 2 and k.

My exemplification concerns the uniform $(2, k)$ generation ($k \geq 3$) of the finite classical simple groups of degree 4.

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Moreover the generators are not uniform.

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Let $\mathbb{F}$ be an algebraically closed field of characteristic p.

We fix $k \geq 3$ such that $(p, k) = 1$, or $k \in \{p, 2p\}$.

We look for $x, y \in \mathrm{SL}_4(\mathbb{F})$ such that

- their projective images have respective orders 2 and k;
- $\langle x, y \rangle$ is irreducible.

The similarity invariants of x can only be

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If $k = 3$ $(p \neq 3)$ the only possibilities for Jordan form of y are:

$$\operatorname{diag}\left(\epsilon, \epsilon, \epsilon^{-1}, \epsilon^{-1}\right) \quad \text{or} \quad \operatorname{diag}\left(1, 1, \epsilon, \epsilon^{-1}\right).$$

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$\epsilon :=$ a primitive k-th root of 1 in $\mathbb{F}$, if $(k,p)=1$.

$\epsilon := 1$ if $k=p$; $\quad \epsilon := -1$ if $k=2p$.

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$$x = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ d & 0 & 0 & 0 \\ 0 & d & 0 & 0 \end{pmatrix}, \; d = \pm 1, \quad y = \begin{pmatrix} 1 & 0 & r_1 & r_2 \\ 0 & 1 & r_3 & r_4 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & \epsilon + \epsilon^{-1} \end{pmatrix}.$$

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$$\dim\ C(x) = 8, \quad \dim\ C(y) = 6 \implies \dim\ C(xy) = 4.$$

$8 + 6 + 4 = 4^2 + 2$. Hence the triple (x, y, xy) is rigid.

Negative results

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Theorem

i) $SL_4(2)$ *is not* $(2,3)$*-generated;*

ii) If $p = 2, 3$*, then* $\mathrm{PSp}_4(q)$ *is not* $(2,3)$*-generated.*

iii) If $p = 2$*, then* $\mathrm{Sp}_4(q)$ *cannot be generated elements having the same similarity invariants as* x *and* y*.*

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i) is known (**Miller**, 1901).
ii) is known (**Liebeck and Shalev**, 1996).
iii) The only infinite class excluded by our choice of y.

For $H = \langle x, y \rangle$ acting on S:

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Hence $d_S^H + \widehat{d_S^H} > 0$. By the rigidity $d_S^H = \widehat{d_S^H} = 1$.

Positive results

For our purposes it is enough to consider generators of shapes:

$$x := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ d & 0 & 0 & 0 \\ 0 & d & 0 & 0 \end{pmatrix}, \quad y := \begin{pmatrix} 1 & 0 & 0 & u \\ 0 & 1 & 0 & r \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & \epsilon + \epsilon^{-1} \end{pmatrix}$$

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Lemma

$\langle x, y \rangle$ is absolutely irreducible, except:
(i) $u = \delta(\epsilon^j - 1)$, $r = \delta(\epsilon^{-j} - 1)$,
(ii) $u + r = \delta(2 - s)$;
(iii) $u = -\epsilon^j r$.

xy has a unique similarity invariant, since:

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$$\begin{aligned}
\mathrm{char}(xy) &= t^4 - drt^3 - d\left(\epsilon + \epsilon^{-1}\right)t^2 - ut + 1; \\
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For a fixed r, it is easy to define u so that (3) holds.

Theorem

Let $0 \neq r \in \mathbb{F}_q$ *and assume* $\mathbb{F}_p\left(r^2, \epsilon + \epsilon^{-1}\right) = \mathbb{F}_q$.
Then, up to a finite number of values of r*:*

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1) if $d = \pm 1$, $u = 0$,

$$\langle x, y \rangle = \mathrm{SL}_4(q).$$

2) if $d = \pm 1$, $u = r^{\sqrt{q}}$, $r \in \mathbb{F}_q \setminus \mathbb{F}_{\sqrt{q}}$,

$$\langle x, y \rangle = \mathrm{SU}_4(q^2).$$

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3) if $d = -1, \quad u = -r$ *and* $p > 2$, $k \neq p, 2p$,

$$\langle x, y \rangle = \mathrm{Sp}_4(q).$$

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$\Omega_4^+ \sim \mathrm{SL}_2(q) \circ \mathrm{SL}_2(q)$
$\Omega_4^- \sim \mathrm{PSL}_2(q^2)$.

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Corollary

Let $3 \leq k$. Assume $k|(q-1)$ or $k|(q+1)$ or $k \in \{p, 2p\}$.

The following groups are $(2,k)$-generated:

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Let $3 \leq k$. Assume $k|(q-1)$ or $k|(q+1)$ or $k \in \{p, 2p\}$.

The following groups are $(2,k)$-generated:

- $\mathrm{SL}_4(q)$ *and* $\mathrm{PSL}_4(q)$;
- $\mathrm{SU}_4(q)$ *and* $\mathrm{PSU}_4(q)$.
- $\mathrm{PSp}_4(q)$, $p > 2$, $k \neq p, 2p$;

The proof is based on the classification of maximal subgroups of the finite classical simple groups of rank 4.

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H. Mitchell (1913),
B.Mwene (1976),
I.Suprunenko and A.Zalesskii (1976),
I.Suprunenko (1981),
P. Kleidman (PHD thesis),
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Sample of proofs

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For $p \geq 5$, the symplectic group $\mathrm{PSp}_4(q)$ has a class of maximal subgroups whose socle is isomorphic to $\mathrm{PSL}_2(q)$. This class arises from the homomorphism

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induced by the action of $\mathrm{SL}_2(q)$ on cubic polynomials in two variables.

By the canonical form of y, (the projective image of) $\langle x, y \rangle$ can lie in a maximal subgroup of this class, only when $k = 3$. In this case:

Lemma

Let $k = 3$, $d = -1$, $u = -r$ and $p \neq 2, 3$. The group $\langle x, y \rangle$ is conjugate to $\phi(H)$ for some $H \leq \mathrm{SL}_2(q)$ if and only if $r^4 = -3$.

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Most of the exceptional values of r arise just when y has order $k = 3$, i.e. for the $(2, 3)$-generation, which is the most difficult.