Symplectic nil-algebras

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Definition

Let F be a field. A symplectic alternating algebra over F is a triple $(V, (\ ,\ ), \cdot)$ where V is a symplectic vector space over F with respect to a non degenerate alternating form $(\ ,\ )$ and $\cdot$ is a bilinear and alternating binary operation on V such that

$$(u \cdot v, w) = (v \cdot w, u)$$

for all $u, v, w \in V$.

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Self-adjoint property

$$(u \cdot x, v) = (x \cdot v, u) = -(v \cdot x, u) = (u, v \cdot x)$$

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Answer to Caranti's question

YES! (Abdollahi, Faghihi, Linton, O'Brien)

Notation

If we have a SAA of dimension $2m$ then we will refer to a basis $x_1, y_1, \cdots, x_m, y_m$ with the property that $(x_i, x_j) = (y_i, y_j) = 0$ and $(x_i, y_j) = \delta_{ij}$ as a standard basis.

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We will also adopt the left-normed convention for multiple products:

$$x_1x_2 \cdots x_n = (\cdots(x_1x_2)\cdots)x_n$$

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Symplectic alternating abelian algebras are nilpotent and SAA which are nilpotent are clearly nil.

Question

?? What can one say about the structure of symplectic nil-algebras??

??Does a symplectic nil-algebra have to be nilpotent ??

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Obviously, every SAA which is associative is also nil-2.

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The following characterization shows that if *char* $F = 2$, then every symplectic nil-2 algebra over F is associative.

Proposition

Let L be a SAA. Then the following are equivalent:

(i) L is nil-2;

(ii) $xyz = -xzy, \ \forall\, x, y, z \in L$;

(iii) $x(yz) = xzy, \ \forall\, x, y, z \in L$.

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$$\Downarrow$$

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$$L \text{ SAA}, \; dim\, L = 6, \; |F| = 3$$

$$L \text{ non abelian nil-algebra} \Leftrightarrow L = O \oplus C_1$$

$$O : \begin{array}{l} x_1 x_2 = 0 \\ x_2 x_3 = 0 \\ x_3 x_1 = 0 \end{array} \qquad C_1 : \begin{array}{l} y_1 y_2 = y_3 \\ y_2 y_3 = 0 \\ y_3 y_1 = 0. \end{array}$$

Proposition

If L is a symplectic nil-2 algebra $dim\, L = 6$, then either L is abelian or isomorphic to $O \oplus C_1$.

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In particular, L is nilpotent of class at most 2.

Theorem

Let L be a SAA over a field F of characteristic $\neq 2$.
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The bounds we have just got are the best possible!

Examples

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Proposition

Let F be a field and L a symplectic nil-2 algebra over F which is nilpotent of class 3.

- If $char\, F = 2$ then $dim\, L \geq 14$;
- if $char\, F \neq 2$ then $dim\, L \geq 8$.

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Let F be a field such that $|F| > 2$ and let L be a symplectic nil-3 algebra over F.
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Proposition

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As a straight consequence of those two results we get $dim\, L \geq 8$ whenever $|F| > 2$.

Let L be a symplectic nil-3 algebra of dimension 8 over a field F of order > 2.

$$L(r) = \begin{pmatrix} a & bx^2 \\ ax & bx \\ ax^2 & b \end{pmatrix} \oplus \begin{pmatrix} x & t \end{pmatrix}.$$

The only non-trivial products not involving x are:

$$\begin{aligned} axa &= rb \\ ax^2a &= -rbx \\ ax^2(ax) &= rbx^2 \\ bxa &= -t \\ axb &= t \end{aligned}$$

$L(r)$ is nil-3 for all $r \in F$ and $L(r)$ is nilpotent of class 5 when $r \neq 0$ and nilpotent of class 3 when $r = 0$.

The number (up to isomorphisms) of symplectic nil-3 algebras of dimension 8 over a field F of order > 2

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$$|F^*/(F^*)^3| + 1$$