Right Engel subgroups

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2. The main structure results
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4. Powerfully embedded Right n-Engel subgroups.

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Question 2. Supposing furthermore that H is upper central. When is the upper central degree bounded in terms of n?

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In fact G is also a non-nilpotent 3-Engel group.

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Here H is upper central right 2-Engel subgroup of degree $m+1$ and G is a nilpotent 3-Engel group of class $m+2$.

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Theorem 4(Burns and Medvedev). There exist positive integers $e(n), c(n)$ such that

$$\gamma_{c_n}(G)^{e(n)} = \{1\}.$$

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The main tool for the proof of Theorem 1 is

Theorem(Zel'manov) Let $L = \langle x_1, \ldots, x_d \rangle$ be a d-generator Lie ring and r, s be positive integers such that

(1)$\sum_{\sigma \in S_n} uv_{\sigma(1)} \cdots v_{\sigma(r)} = 0$ for all $u, v_1, \ldots, v_r \in L$
(2) $uv^s = 0$ for all $u \in L$ and all Lie products v in the generators.

Then L is nilpotent of (d, r, s)-bounded class.

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$$[H,_c G] \leq [H,_{c+1} G] \Rightarrow [H,_c G] = \{1\}.$$

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Need to work with $H_i = \sqrt[H]{[H,_i G]}$ and $G_i = \sqrt[G]{\gamma_i(G)}$ instead.

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$$[H,_{r(n)+c(n)} G] \leq [H^{p^{r(n)}},_{c(n)} G] = \{1\}.$$