The Brauer-Clifford group of G-rings

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Notation

G a finite group,

$\mathrm{Irr}(G)$ its complex irreducible characters.

p a prime.

F a field.

If $\chi \in \mathrm{Irr}(G)$ then $F(\chi)$ is the field of values of χ over F.

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Applications of the Brauer-Clifford group

Theorem

Let P be a p-subgroup of G, and let N be a normal p'-subgroup of G, and suppose that PN is a normal subgroup of G. Let $\theta \in \mathrm{Irr}(N)$ be P-invariant, and let $\phi \in \mathrm{Irr}(\mathrm{C}_N(P))$ be its Glauberman correspondent. Then the Clifford theory of θ in G over $\mathbf{Q}_p$ is the same as the Clifford theory of ϕ in $\mathrm{N}_G(P)$ over $\mathbf{Q}_p$.

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Strengthened McKay Conjecture (Alperin, Isaacs, Navarro, Turull)

There is a bijection $f : \mathrm{Irr}_{p'}(G) \to \mathrm{Irr}_{p'}(\mathrm{N}_G(P))$ such that

$$f(\chi)(1) \equiv \pm\chi(1) \pmod{p} \quad \text{for all } \chi \in \mathrm{Irr}_{p'}(G)$$

$$\mathbf{Q}_p(f(\chi)) = \mathbf{Q}_p(\chi) \quad \text{for all } \chi \in \mathrm{Irr}_{p'}(G)$$

$$\mathrm{m}_{\mathbf{Q}_p}(f(\chi)) = \mathrm{m}_{\mathbf{Q}_p}(\chi) \quad \text{for all } \chi \in \mathrm{Irr}_{p'}(G),$$

and similar conditions on f for the block version.

Theorem

If G is any p-solvable group, then the strengthened McKay Conjecture holds for G.

Modules over G-rings

Definition

A G-ring is a ring R together with a group homomorphism $\phi : G \to \mathrm{Aut}(R)$.

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Lemma

Let Z be a commutative simple G-ring. Let $e_1, \ldots, e_\alpha$ be the primitive idempotents of Z, and set $K_i = e_i Z$ Then,

1. *K_i is a field.*
2. *G acts transitively on $\{e_1, ..., e_\alpha\}$.*
3. *$Z = K_1 \oplus \cdots \oplus K_\alpha$.*

Definition

Let R be a G-ring. Then the group ring RG is the set of all formal linear combinations of G with coefficients in R.

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Definition

Let G be a finite group and R be a G-ring. A G-module over R is simply an RG-module.

Theorem

Let Z be a commutative simple G-ring, and let M be a G-module over Z. Then M is free as a Z-module.

Definition

Let Z be a commutative G-ring. A G-algebra over Z is a G-ring A together with an additional structure on A of G-module over Z, which uses the algebra addition on A, and satisfies the conditions that, for all $a, b \in A$, $w, z \in Z$, and $g \in G$, we have $(wa)(zb) = (wz)(ab)$, and ${}^g(wa) = {}^g w \, {}^g a$.

Definition

Let Z be a commutative G-ring, let A be a G-algebra over Z, and let $u : Z \to \mathrm{Z}(A)$ be the structural G-ring homomorphism. We will say that A is a central G-algebra over Z if u is an isomorphism from Z onto $\mathrm{Z}(A)$.

Theorem

Let G be a finite group, and Z be a commutative simple G-ring. Let A be a central simple G-algebra of finite rank over Z. Then, for each primitive idempotent e of Z, we have that eA is a central simple algebra of finite dimension over the field eZ. Furthermore, if $e_1, \ldots, e_\alpha$ are the primitive idempotents of Z, then

$$A = e_1 A \oplus \cdots \oplus e_\alpha A,$$

as rings.

Theorem

Let Z be a commutative simple G-ring. Let A and B be central simple G-algebras of finite rank over Z. Then $A \otimes_Z B$ is a central simple G-algebra of finite rank over Z.

Definition

Let Z a commutative simple G-ring. We say that a central simple G-algebra A over Z is trivial if there exists a finitely generated non-zero G-module M over Z such that $\mathrm{End}_Z(M)$ is isomorphic to A as central simple G-algebras over Z.

Lemma

Let Z a commutative simple G-ring, and let T and S be trivial central simple G-algebras over Z. Then, $T \otimes_Z S$ is a trivial central simple G-algebra over Z.

Lemma

Let Z a commutative simple G-ring. Let A be a central simple G-algebra of finite rank over Z. Then, A^{op} is a central simple G-algebra of finite rank over Z, and $A \otimes_Z A^{op} \simeq \mathrm{End}_Z(A)$, as G-algebras over Z, where we view A as a finitely generated G-module over Z. In particular, $A \otimes_Z A^{op}$ is a trivial G-algebra over Z.

Let Z be a commutative simple G-ring.

Definition

Suppose A, and B are central simple G-algebras of finite rank over Z. Then, we say that A is equivalent to B if and only if there exist trivial G-algebras T_1 and T_2 over Z such that

$$A \otimes_Z T_1 \simeq B \otimes_Z T_2$$

as central G-algebras over Z.

Definition

We define the Brauer-Clifford group of G over Z to be the set

$$\mathrm{BrClif}(G, Z)$$

of equivalence classes of central simple G-algebras of finite rank over Z, together with the binary operation induced by the tensor product over Z of central simple G-algebras over Z.

Theorem

Let Z be a commutative simple G-algebra. Then, the Brauer-Clifford group $\mathrm{BrClif}(G, Z)$ *of G over Z is an abelian group.*

Further notation

Let Z be a commutative simple G-ring.

Let e_1 be a primitive idempotent of Z, and set

$$K_1 = e_1 Z,\ I_1 = \mathrm{C}_G(e_1),\text{ and } F_1 = K_1^{I_1}.$$

Then, K_1 is an extension field of the field F_1,

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Lemma

Let A be a central simple G-algebra of finite rank over Z. Then, $e_1 A$ is a central simple algebra over the field K_1, and its class $[e_1 A]$ in the Brauer group $\mathrm{Br}(K_1)$ is invariant under the action of I_1, we write $[e_1 A] \in \mathrm{Br}(K_1)^{I_1}$. Furthermore, the map

$$\phi : \mathrm{BrClif}(G, Z) \to \mathrm{Br}(K_1)^{I_1}$$

defined by $\phi([[A]]) = [e_1 A]$, for all central simple G-algebra A over Z (where $[[A]]$ is the class in $\mathrm{BrClif}(G, Z)$ of A), is a group homomorphism. Finally, the kernel of ϕ does not depend on the choice of the idempotent e_1.

Definition

Let G be a finite group, and Z be a commutative simple G-ring. We call the kernel of any one of the homomorphisms ϕ the full matrix subgroup of the Brauer-Clifford group of G over Z, and we denote it by $\mathrm{FMBrClif}(G, Z)$.

Theorem

Let G be a finite group, and Z be a commutative simple G-ring. Then $\mathrm{FMBrClif}(G, Z)$ is isomorphic to $H^2(G, Z^\times)$.

Clifford theory

Set up

$\pi : G \to \overline{G}$ a surjective homomorphism of finite groups

$H = \ker(\pi)$ and F a field.

S an irreducible FG-module.

Definition

Set $Z_0 = \mathrm{Z}(FH)/J(\mathrm{Z}(FH))$, and $\overline{G}$ acts on Z_0. Furthermore, $\mathrm{Z}(FH)/J(\mathrm{Z}(FH))$ acts on S. Let e be the unique primitive idempotent of $Z_0^{\overline{G}}$ which acts non-trivially on S. Then, we set $Z = eZ_0$. We define the center ring of S with respect to S and F to be the $\overline{G}$-ring Z, and we denote it by $\mathrm{Z}(S, \pi, F)$.

Lemma

$Z(S, \pi, F)$ is a commutative simple $\overline{G}$-ring.

Definition

An FG-module M is S-quasihomogeneous (with respect to H) if it is not 0, its restriction to H is completely reducible, and e acts as the identity on M.

Theorem

Suppose that M is any S-quasihomogeneous G-module over F. Then, $\mathrm{End}_{FH}(M)$ is a central simple $\overline{G}$-algebra over Z. Furthermore, the class in $\mathrm{BrClif}(\overline{G}, Z)$ of $\mathrm{End}_{FH}(M)$ does not depend on the choice of M.

This assigns an element $[[S]] \in \mathrm{BrClif}(\overline{G}, Z)$ to S.

Theorem

The element $[[S]]$ determines the Clifford theory of S.

Clifford theory in the classical cases

Theorem (Clifford theory in the induced case)

Let H be a normal subgroup of G, and let $\theta \in \mathrm{Irr}(H)$ Let I be the inertia group of θ. Then induction provides the Clifford theory of θ over G from that of θ over I.

The center ring

Let $\theta_1 = \theta, \ldots, \theta_n$ be the G-conjugates of θ.

Let $e_{\theta_1}, \ldots, e_{\theta_n}$ be the corresponding idempotents of $Z(\mathbf{C}H)$.

Let $e = e_{\theta_1} + \cdots + e_{\theta_n}$.

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$Z(\theta, \pi, \mathbf{C}) \simeq \mathbf{C} \oplus \cdots \oplus \mathbf{C}$ where there are n copies of $\mathbf{C}$,

and G/H acts transitively on these copies.

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Theorem (Brauer-Clifford in the induced case)

Let H be a normal subgroup of G, let $\theta \in \mathrm{Irr}(H)$, and let I be the inertia group of θ. Then

$$\mathrm{Ind}_{I/H}^{G/H} : \mathrm{BrClif}(I/H, \mathrm{Z}(\theta, \pi_I, \mathbf{C})) \to \mathrm{BrClif}(G/H, \mathrm{Z}(\theta, \pi, \mathbf{C}))$$

is a group isomorphism, and $\mathrm{Ind}_{I/H}^{G/H}([[\theta]]_{I,\mathbf{C}}) = [[\theta]]_{G,\mathbf{C}}$.

Theorem (Clifford theory in the homogeneous case)

Let H be a normal subgroup of G, let $\theta \in \mathrm{Irr}(H)$, and suppose that θ is G-invariant. Then the Clifford theory of θ over G is determined by some element of $H^2(G/H, \mathbf{C}^\times)$.

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$$\mathrm{BrClif}(G/H, \mathbf{C}) \simeq H^2(G/H, \mathbf{C}^\times).$$

Theorem (Clifford theory in the $G = H$ case)

Suppose that $H = G$, and let $\theta \in \mathrm{Irr}(H)$. Then the Schur index of θ over any extension field of $\mathbf{Q}(\theta)$ is determined by some element of $\mathrm{Br}(\mathbf{Q}(\theta))$.

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Suppose that $H = G$, and let $\theta \in \mathrm{Irr}(H)$. Then $\mathrm{Z}(\theta, \pi, \mathbf{Q}) \simeq \mathbf{Q}(\theta)$, and

$$\mathrm{BrClif}(G/G, \mathbf{Q}(\theta)) \simeq \mathrm{Br}(\mathbf{Q}(\theta)).$$