The Brauer-Clifford group of $G$-rings

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Notation

$G$ a finite group,
$Irr(G)$ its complex irreducible characters.
$p$ a prime.

$F$ a field.

If $\chi \in Irr(G)$ then $F(\chi)$ is the field of values of $\chi$ over $F$. 
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Applications of the Brauer-Clifford group
Theorem

Let $P$ be a $p$-subgroup of $G$, and let $N$ be a normal $p'$-subgroup of $G$, and suppose that $PN$ is a normal subgroup of $G$. Let $\theta \in \text{Irr}(N)$ be $P$-invariant, and let $\phi \in \text{Irr}(C_N(P))$ be its Glauberman correspondent. Then the Clifford theory of $\theta$ in $G$ over $\mathbb{Q}_p$ is the same as the Clifford theory of $\phi$ in $N_G(P)$ over $\mathbb{Q}_p$. 
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**Strengthened McKay Conjecture (Alperin, Isaacs, Navarro, Turull)**

There is a bijection $f : \text{Irr}_{p'}(G) \rightarrow \text{Irr}_{p'}(N_G(P))$ such that

$$f(\chi)(1) \equiv \pm \chi(1) \pmod{p} \quad \text{for all } \chi \in \text{Irr}_{p'}(G)$$

$$Q_p(f(\chi)) = Q_p(\chi) \quad \text{for all } \chi \in \text{Irr}_{p'}(G)$$

$$m_{Q_p}(f(\chi)) = m_{Q_p}(\chi) \quad \text{for all } \chi \in \text{Irr}_{p'}(G),$$

and similar conditions on $f$ for the block version.
Theorem

If $G$ is any $p$-solvable group, then the strengthened McKay Conjecture holds for $G$. 
Modules over $G$-rings
Definition

A *G*-ring is a ring $R$ together with a group homomorphism $\phi : G \to \text{Aut}(R)$.

Definition

$R$ is a simple *G*-ring if $R$ is not zero and it has no non-trivial proper two-sided $G$-invariant ideals.
Definition

A $G$-ring is a ring $R$ together with a group homomorphism $\phi : G \to \text{Aut}(R)$.

Definition

$R$ is a simple $G$-ring if $R$ is not zero and it has no non-trivial proper two-sided $G$-invariant ideals.
Lemma

Let $Z$ be a commutative simple $G$-ring. Let $e_1, \ldots, e_\alpha$ be the primitive idempotents of $Z$, and set $K_i = e_i Z$. Then,

1. $K_i$ is a field.
2. $G$ acts transitively on \{e_1, \ldots, e_\alpha\}.
3. $Z = K_1 \oplus \cdots \oplus K_\alpha$. 
Definition

Let $R$ be a $G$-ring. Then the group ring $RG$ is the set of all formal linear combinations of $G$ with coefficients in $R$.

Even when $R = \mathbb{Z}$ is commutative, a group ring is not necessarily an algebra over $\mathbb{Z}$. 
Definition
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Even when $R = Z$ is commutative, a group ring is not necessarily an algebra over $Z$. 
Definition

Let $G$ be a finite group and $R$ be a $G$-ring. A $G$-module over $R$ is simply an $RG$-module.
Theorem

Let $Z$ be a commutative simple $G$-ring, and let $M$ be a $G$-module over $Z$. Then $M$ is free as a $Z$-module.
Definition

Let $Z$ be a commutative $G$-ring. A $G$-algebra over $Z$ is a $G$-ring $A$ together with an additional structure on $A$ of $G$-module over $Z$, which uses the algebra addition on $A$, and satisfies the conditions that, for all $a, b \in A$, $w, z \in Z$, and $g \in G$, we have $(wa)(zb) = (wz)(ab)$, and $g(wa) = gwg a$. 
Definition

Let $Z$ be a commutative $G$-ring, let $A$ be a $G$-algebra over $Z$, and let $u : Z \rightarrow Z(A)$ be the structural $G$-ring homomorphism. We will say that $A$ is a \textit{central} $G$-algebra over $Z$ if $u$ is an isomorphism from $Z$ onto $Z(A)$. 
Theorem

Let $G$ be a finite group, and $Z$ be a commutative simple $G$-ring. Let $A$ be a central simple $G$-algebra of finite rank over $Z$. Then, for each primitive idempotent $e$ of $Z$, we have that $eA$ is a central simple algebra of finite dimension over the field $eZ$. Furthermore, if $e_1, \ldots, e_\alpha$ are the primitive idempotents of $Z$, then

$$A = e_1 A \oplus \cdots \oplus e_\alpha A,$$

as rings.
Theorem

Let $Z$ be a commutative simple $G$-ring. Let $A$ and $B$ be central simple $G$-algebras of finite rank over $Z$. Then $A \otimes_Z B$ is a central simple $G$-algebra of finite rank over $Z$. 

Definition

Let $Z$ a commutative simple $G$-ring. We say that a central simple $G$-algebra $A$ over $Z$ is trivial if there exists a finitely generated non-zero $G$-module $M$ over $Z$ such that $\text{End}_Z(M)$ is isomorphic to $A$ as central simple $G$-algebras over $Z$. 
Lemma

Let $Z$ a commutative simple $G$-ring, and let $T$ and $S$ be trivial central simple $G$-algebras over $Z$. Then, $T \otimes_Z S$ is a trivial central simple $G$-algebra over $Z$. 
Lemma

Let $Z$ a commutative simple $G$-ring. Let $A$ be a central simple $G$-algebra of finite rank over $Z$. Then, $A^{\text{op}}$ is a central simple $G$-algebra of finite rank over $Z$, and $A \otimes_Z A^{\text{op}} \cong \text{End}_Z(A)$, as $G$-algebras over $Z$, where we view $A$ as a finitely generated $G$-module over $Z$. In particular, $A \otimes_Z A^{\text{op}}$ is a trivial $G$-algebra over $Z$. 
Let $Z$ be a commutative simple $G$-ring.

**Definition**

Suppose $A$, and $B$ are central simple $G$-algebras of finite rank over $Z$. Then, we say that $A$ is equivalent to $B$ if and only if there exist trivial $G$-algebras $T_1$ and $T_2$ over $Z$ such that

$$A \otimes_Z T_1 \sim B \otimes_Z T_2$$

as central $G$-algebras over $Z$. 
We define the Brauer-Clifford group of $G$ over $\mathbb{Z}$ to be the set

$$\text{BrClif}(G, \mathbb{Z})$$

of equivalence classes of central simple $G$-algebras of finite rank over $\mathbb{Z}$, together with the binary operation induced by the tensor product over $\mathbb{Z}$ of central simple $G$-algebras over $\mathbb{Z}$.
Let $Z$ be a commutative simple $G$-algebra. Then, the Brauer-Clifford group $\text{BrClif}(G, Z)$ of $G$ over $Z$ is an abelian group.
Further notation

Let $Z$ be a commutative simple $G$-ring.
Let $e_1$ be a primitive idempotent of $Z$, and set

$$K_1 = e_1 Z, \quad I_1 = C_G(e_1), \quad \text{and} \quad F_1 = K_1^{I_1}.$$ 

Then, $K_1$ is an extension field of the field $F_1$, and $K_1/F_1$ is a finite Galois extension.
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Lemma

Let $A$ be a central simple $G$-algebra of finite rank over $\mathbb{Z}$. Then, $e_1 A$ is a central simple algebra over the field $K_1$, and its class $[e_1 A]$ in the Brauer group $\text{Br}(K_1)$ is invariant under the action of $I_1$, we write $[e_1 A] \in \text{Br}(K_1)^I$. Furthermore, the map

$$\phi : \text{BrClif}(G, \mathbb{Z}) \rightarrow \text{Br}(K_1)^I$$

defined by $\phi([[A]]) = [e_1 A]$, for all central simple $G$-algebra $A$ over $\mathbb{Z}$ (where $[[A]]$ is the class in $\text{BrClif}(G, \mathbb{Z})$ of $A$), is a group homomorphism. Finally, the kernel of $\phi$ does not depend on the choice of the idempotent $e_1$. 

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Brauer-Clifford

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Definition
Let $G$ be a finite group, and $Z$ be a commutative simple $G$-ring. We call the kernel of any one of the homomorphisms $\phi$ the full matrix subgroup of the Brauer-Clifford group of $G$ over $Z$, and we denote it by $\text{FMBrClif}(G, Z)$. 
Theorem

Let $G$ be a finite group, and $Z$ be a commutative simple $G$-ring. Then $\text{FMBrClif}(G, Z)$ is isomorphic to $H^2(G, Z^\times)$. 
Clifford theory
Set up

\[ \pi : G \rightarrow \overline{G} \text{ a surjective homomorphism of finite groups} \]

\[ H = \ker(\pi) \text{ and } F \text{ a field.} \]

\[ S \text{ an irreducible } FG\text{-module.} \]
Definition

Set $Z_0 = Z(FH)/J(Z(FH))$, and $\bar{G}$ acts on $Z_0$. Furthermore, $Z(FH)/J(Z(FH))$ acts on $S$. Let $e$ be the unique primitive idempotent of $Z_0^{\bar{G}}$ which acts non-trivially on $S$. Then, we set $Z = eZ_0$. We define the center ring of $S$ with respect to $S$ and $F$ to be the $\bar{G}$-ring $Z$, and we denote it by $Z(S, \pi, F)$. 
Lemma

$Z(S, \pi, F)$ is a commutative simple $\overline{G}$-ring.
Definition

An $FG$-module $M$ is $S$-quasihomogeneous (with respect to $H$) if it is not 0, its restriction to $H$ is completely reducible, and $e$ acts as the identity on $M$. 
Theorem

Suppose that $M$ is any $S$-quasihomogeneous $G$-module over $F$. Then, $\text{End}_{FH}(M)$ is a central simple $\overline{G}$-algebra over $\mathbb{Z}$. Furthermore, the class in $\text{BrClif}(\overline{G}, \mathbb{Z})$ of $\text{End}_{FH}(M)$ does not depend on the choice of $M$.

This assigns an element $[[S]] \in \text{BrClif}(\overline{G}, \mathbb{Z})$ to $S$. 
Theorem

The element $[[S]]$ determines the Clifford theory of $S$. 
Clifford theory in the classical cases
Theorem (Clifford theory in the induced case)

Let \( H \) be a normal subgroup of \( G \), and let \( \theta \in \text{Irr}(H) \). Let \( I \) be the inertia group of \( \theta \). Then induction provides the Clifford theory of \( \theta \) over \( G \) from that of \( \theta \) over \( I \).
The center ring

Let $\theta_1 = \theta, \ldots, \theta_n$ be the $G$-conjugates of $\theta$.

Let $e_{\theta_1}, \ldots, e_{\theta_n}$ be the corresponding idempotents of $Z(CH)$.

Let $e = e_{\theta_1} + \cdots + e_{\theta_n}$.
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Then $Z(\theta, \pi, C) = eZ(CH)$ as a $G/H$-ring.

$Z(\theta, \pi, C) \simeq C \oplus \cdots \oplus C$ where there are $n$ copies of $C$, and $G/H$ acts transitively on these copies.
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This transitive action determines the inertia group $I$ up to $G$-conjugacy.
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The classical cases

Theorem (Clifford theory in the induced case)

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Let $H$ be a normal subgroup of $G$, let $\theta \in \text{Irr}(H)$, and let $I$ be the inertia group of $\theta$. Then

$$\text{Ind}_{I/H}^{G/H} : \text{BrClif}(I/H, Z(\theta, \pi_I, C)) \to \text{BrClif}(G/H, Z(\theta, \pi, C))$$

is a group isomorphism, and $\text{Ind}_{I/H}^{G/H} ([[\theta]]_{I,C}) = [[\theta]]_{G,C}$. 
Theorem (Clifford theory in the homogeneous case)

Let $H$ be a normal subgroup of $G$, let $\theta \in \text{Irr}(H)$, and suppose that $\theta$ is $G$-invariant. Then the Clifford theory of $\theta$ over $G$ is determined by some element of $H^2(G/H, \mathbb{C}^\times)$. 
The classical cases

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$$\text{BrClif}(G/H, \mathbb{C}) \cong H^2(G/H, \mathbb{C}^\times).$$
The classical cases

Theorem (Clifford theory in the $G = H$ case)

Suppose that $H = G$, and let $\theta \in \text{Irr}(H)$. Then the Schur index of $\theta$ over any extension field of $\mathbb{Q}(\theta)$ is determined by some element of $\text{Br}(\mathbb{Q}(\theta))$. 
The classical cases

Theorem (Clifford theory in the $G = H$ case)

Suppose that $H = G$, and let $\theta \in \text{Irr}(H)$. Then the Schur index of $\theta$ over any extension field of $\mathbb{Q}(\theta)$ is determined by some element of $\text{Br}(\mathbb{Q}(\theta))$.

Theorem (Brauer-Clifford in the $G = H$ case)

Suppose that $H = G$, and let $\theta \in \text{Irr}(H)$. Then $Z(\theta, \pi, \mathbb{Q}) \simeq \mathbb{Q}(\theta)$, and

$$\text{BrClif}(G/G, \mathbb{Q}(\theta)) \simeq \text{Br}(\mathbb{Q}(\theta)).$$