Decomposition numbers for projective modules of finite Chevalley groups

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Let G be a finite group and p a prime number. Representations of G over an algebraically closed field of characteristic p are called p-**modular**, and those over the complex numbers are called **ordinary**. An ordinary representation is equivalent to a representation ϕ over a finite extension of the rationals, and moreover, over a maximal subring R of the latter field not containing p. In addition, R has a unique maximal ideal I such that $F = R/I$ is a finite field of characteristic p. So if $\phi(G) \subseteq GL(n, R)$ for some n, then the representation $\overline{\phi} : G \to GL(n, F)$ obtained from the natural projection $GL(n, R) \to GL(n, F)$ is called the reduction of ϕ modulo p. If ϕ is irreducible then $\overline{\phi}$ is not always irreducible. Let $\overline{F}$ be the algebraic closure of F. The multiplicities of irreducible representations of G over $\overline{F}$ as constituents of $\overline{\phi}$ are called the **decomposition numbers**. They can be arranged in shape of a matrix, called the **decomposition matrix**.

Rows are labeled by the ordinary irreducible characters and the columns are by the p-modular ones. So the decomposition numbers of an ordinary representation are in the respective row. The matrix depends on the ordering the irreducible representations, both ordinary and modular. If p is coprime to the order of G then for a suitable ordering the decomposition matrix is the identity matrix. Below is the decomposition matrix for $G = PSL(3,3)$ and $p = 3$.

	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6	ϕ_7	ϕ_8	ϕ_9
χ_1	1								
χ_2				1	1				
χ_3		1	1			1			
χ_4	1						1		
χ_5	1							1	
χ_6			1		1	1			
χ_7		1		1		1			
χ_8	1	1	1	1	1	1			
χ_9	1	1				1		1	
χ_{10}	1		1			1	1		
χ_{11}	2					1	1	1	
St									1

(The empty positions are zeros.) If ith column has all but one zeros then ϕ_i and corresponding χ_j are said to be **of defect 0**. Symbol St denotes the Steinberg representation character, the only one of defect 0.

For applications it is important to know the decomposition matrices for various finite groups. The theory of modular representations develops methods for understanding properties of the matrix. Brauer's theory of blocks is one of the most powerful tool of the theory. However, this does not help for groups such as $PSL(d,p)$ as in this case there are only two Brauer blocks. There is another approach, via principal indecomposable $\overline{F}G$-modules, also developed by Brauer. Some recent results I explain today demonstrate that this approach is rather efficient and has a lot of potential when G is a Chevalley group in defining characteristic p.

Principal indecomposable modules (PIM's) are the direct summands of the regular $\overline{F}G$-module. The remarkable feature of PIM's is that they can be obtained by the above reduction from RG-modules of the same dimension, and the latter can be decomposed as a direct sum of ordinary representations. It is rather surprizing that the multiplicities of the ordinary irreducible representations in the decomposition of a PIM coincide with some column of the decomposition matrix. Moreover, all columns can be obtained in this way. These are classical results which are valid for any G. There is no hint that in general it is easier to compute the decomposition matrix via PIM's. But for groups of Lie type in characteristic p this seems to be true.

What is helpful in this case is that there is a PIM with well-studied properties, namely, the Steinberg PIM. I denote by St. It lifts to an irreducible RG-module. The dimension of St is lower than that of any other PIM. The tensor product of St with any $\overline{F}G$-module is a direct sum of PIM's and every PIM occurs in a suitable tensor product. Ballard, Humphreys, Jantzen and other studied PIM's via this approach. See Humphreys' survey in Bull. AMS 16(1987), 247 - 263. I illustrate this with a recent result in Hiss-Z, Repres. theory 13(2009), 427-459:

Theorem 1. Let $G = Sp(2n, q)$, q odd, the the symplectic group. Then there exists an FG-module V of dimension q^n such that $V \otimes St$ is multiplicity free, that is, every irreducible representation occurs in the decomposition with multiplicity at most 1.

Corollary. Let P be a PIM occurring in the decomposition of $V \otimes St$. Then the respective column of the decomposition matrix is a $(0, 1)$-column.

For brevity, a column of the decomposition matrix with entries 0,1 are called a $(0, 1)$-column.

The number of the PIM's in the corollary is at least $q^n/n! \cdot 2^n$. So this is a lower bound for the number of $(0, 1)$-columns in the decomposition matrix of $Sp(2n, q)$, q odd, but this is a rather rough bound.

Problem. Provide a reasonable lower bound for the number of $(0, 1)$-columns for arbitrary Chevalley groups.

I do not provide here an explicit definition of Chevalley groups, recalling instead that general linear, unitary, symplectic and orthogonal groups over finite fields are examples are of Chevalley groups. In general, Chevalley groups are constructed as automorphism groups of simple Lie algebras obtained by taking a finite field for the ground field (in place of the complex number field).

There is an advantage to deal with characters rather than with representations. The knowledge of the character of an RG-module is sufficient to decompose it as a direct sum of irreducible ones.

A function $\phi : G \to \mathbf{C}$ is called a **generalized character** (or virtual character) if ϕ is an integer linear combination of irreducible characters. If $\phi = \sum m_i \chi_i$, where χ_i are irreducible characters then m_i is called the **multiplicity** of χ_i in ϕ and the χ_i's with $m_i \neq 0$ are called **irreducible constituents of** ϕ. If every $m_i = 1$ or 0, one says that ϕ is **multiplicity free**.

The character of the RG-module corresponding to a PIM is the product of a suitable generalized character with the Steinberg character (Ballard-Lusztig).

Problem. When is the converse true?

This is a difficult question but there are sufficient conditions found by Ballard. I shall not comment these but instead I provide sufficient conditions for $\phi \cdot St$ to be multiplicity free. (Now St stands for the Steinberg character.)

In fact, the above products $\phi \cdot St$ were first considered within the study of PIM's.

Ballard (1978) proves that $\phi \cdot St$ in many instances is a character of a PIM, and in more cases this is a character of a direct sum of PIM's. Therefore, if one can decompose $\phi \cdot St$ as a sum of irreducible characters of G, one contributes to the knowledge of the decomposition numbers.

Problem. When $\phi \cdot St$ is a character of a PIM, and when $\phi \cdot St$ is a character of a direct sum of PIM's? Compute the direct summands whenever it is possible.

A few cases not contained in Ballard's paper have been considered by Tsushima (1990). There is further potential in this direction.

Decomposition numbers

There are important abelian subgroups in Chevalley groups called maximal tori. Usually these are just maximal abelian p'-subgroups of G but there are exceptions.

We call a generalized character ϕ a **T-character** if for every maximal torus T of G the restriction $\phi|_T$ is multiplicity free. The result below are from a joint paper with G. Hiss (Proc. AMS 138(2010), 1907-1921.)

Theorem 2. Let G be a Chevalley group of defining characteristic p, and let ϕ be a T-character. Then $\phi \cdot St$ is a proper character, which is the sum of exactly $\phi(1)$ distinct irreducible characters.

The first assertion provides a rather nice sufficient condition for $\phi \cdot St$ to be a proper character. In addition, $\phi \cdot St$ is multiplicity free as well.

It does not follow from Theorem 2 that $\phi \cdot St$ is a character of a sum of PIM's. Some results are due to Ballard but it is not clear whether every T-character satisfies Ballard's assumption. Nonetheless this is rather often the case so this theorem is a source of many $(0, 1)$-columns in the decomposition matrix. One has to study the problem further in order to make a good lower bound for the number of the $(0, 1)$-columns.

Whenever this is true, Theorem 2 tells us that the decomposition numbers for $\phi \cdot St$ are 1 or 0, and there are exactly $\phi(1)$ occurrences of 1.

The method used in the proof of Theorem 2 allows to identify the irreducible characters χ_i of G occurring in the the decomposition $\phi \cdot St = \sum_i \chi_i$. It is turns out that they are so called regular characters introduced by Deligne and Lusztig. This helps to identify the positions with entry 1 at the column arising in Theorem 2.

I wish to close this discussion with the following elementary result which has been inspired by the above consideration.

Recall that the conjugacy classes of the maximal tori in $GL(n,q)$ are in bijection with the conjugacy classes of the symmetric group S_n. Let $G = GL(n,2)$, and for $w \in S_n$ let T_w be a maximal torus in G corresponding to a conjugacy class of w in S_n. Let V_k be the k-th exterior power of the natural G-module V. Then the dimension of the fixed point space of T_w on V_k is equal to the value at w of the induced character $(1_{S_k \times S_{n-k}})^{S_n}$. Here $1_{S_k \times S_{n-k}}$ means the trivial character of the subgroup $S_k \times S_{n-k}$ and $()^{S_n}$ denotes the induction.