Fixed points of coprime automorphisms of finite groups

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Ischia Group Theory 2012

Conselho Nacional de Desenvolvimento Científico e Tecnológico

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- For us A will be an elementary abelian group of order q^r with $r \geq 2$ acting on a finite q'-group G, where q is a prime.

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- Quantitative version for finite p-groups of a Lazard's criterion for a pro-p group to be p-adic analytic;
- Theorem of Bahturin and Zaicev on Lie algebras admitting a group of automorphisms whose fixed-point subalgebra satisfies a polynomial identity.

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- Since $L_p(G)$ is nilpotent, it follows that G has a powerful subgroup of $\{q, m\}$-bounded index.
- The proof in the case where G is a powerful p-group is straightforward.

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- The result is true for finite p-groups, so the exponents of Sylow subgroups of G will be $\{q, m\}$-bounded and also the exponent of G.

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Theorem (Guralnick-Shumyatsky, 2001)

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- The assumption that $|A| = q^3$ is essential here and the theorem fails if $|A| = q^2$.

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But the reduction to p-groups is much more complicated. We need the following result.

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Theorem (generation result)

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- There is a bound (depending only on q and m) on the exponent of $P \cap G'$ for an A-invariant Sylow p-subgroup P of G for each prime p.

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So the idea of Guralnick and Shumyatsky is not quite adequate: for G' the situation was easier since $(G' \cap Q)/Q'$ is abelian, for a Sylow p-subgroup Q of G.

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- - degree 1: $[C_G(A_i), C_G(A_j)] \cap C_G(A_l), \quad i, j, l \leq s$

Some properties of A-special subgroups

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- Let R_k be the subgroup generated by all A-special subgroups of G of degree k. Then $R_k = G^{(k)}$.
- If $2^k \leq r-1$ and H is an A-special subgroup of G of degree k, then H is contained in the kth derived group of $C_G(B)$ for some subgroup $B \leq A$ such that $|A/B| \leq q^{2^k}$.

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- For all $l \geq 1$ the lth derived group $P^{(l)}$ is the product of the subgroups of the form $P^{(l)} \cap P_j$, where $j = 1, \ldots, t$.

Main result

Theorem

Let q be a prime, m a positive integer and A an elementary abelian group of order q^r with $r \geq 2$ acting on a finite q'-group G. If, for some integer d such that $2^d \leq r-1$, the dth derived group of $C_G(a)$ has exponent dividing m for any $a \in A^{\#}$, then the dth derived group $G^{(d)}$ has $\{m, q, r\}$-bounded exponent.

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Some Bibliography

More details in

- C. Acciarri and P. Shumyatsky, ‘Fixed points of coprime operator groups’, *J. Algebra* **342** (2011), 161–174.
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