Products of Locally Supersoluble Groups

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Ischia Group Theory

Definition

A group G is the *product* of two subgroups A and B if

$$G = AB = \{ab \mid a \in A, b \in B\}.$$

In this case, we also say that G is *factorized* by A and B.

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Theorem (N. Itô, 1955)

Let the group $G = AB$ be the product of two abelian subgroups A and B. Then G is metabelian.

Let $G = AB$ be a group factorized by two subgroups A and B. If N is a normal subgroup of G, then the subgroup

$$AN \cap BN$$

has an interesting triple factorization:

$$AN \cap BN = (A \cap BN)N = (B \cap AN)N = (A \cap BN)(B \cap AN).$$

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The structure of normal subgroups of factorized groups leads to the consideration of triply factorized groups

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The structure of normal subgroups of factorized groups leads to the consideration of triply factorized groups

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where K is a normal subgroup of G.

Actually, even without the normality assumption, many properties of a group $G = AB = AC = BC$ can be detected from the structure of the subgroups A, B and C.

Theorem (O.H. Kegel, 1965)

Let the finite group

$$G = AB = AC = BC$$

be the product of two nilpotent subgroups A and B and a supersoluble subgroup C. Then G is supersoluble.

Recall that a *formation* is a class $\mathfrak{F}$ of finite groups such that every homomorphic image of an $\mathfrak{F}$-group is an $\mathfrak{F}$-group and if G/N and G/M are $\mathfrak{F}$-groups, then $G/N \cap M$ also belongs to $\mathfrak{F}$. Moreover, the formation $\mathfrak{F}$ is *saturated* if the finite group G belongs to $\mathfrak{F}$ whenever the Frattini factor group $G/\Phi(G)$ is in $\mathfrak{F}$.

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Theorem (F.G. Peterson, 1973)

Let $\mathfrak{F}$ be a saturated formation containing all finite nilpotent groups, and let the finite group

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Peterson also produced an example to show that a finite group $G = AB = AC = BC$, factorized by a nilpotent subgroup A and two supersoluble subgroups B and C, need not be supersoluble.

Theorem (R. Baer)

If G is a finite group with nilpotent commutator subgroup, then every collection of supersoluble normal subgroups of G generates a supersoluble subgroup.

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If G is a finite group with nilpotent commutator subgroup, then every collection of supersoluble normal subgroups of G generates a supersoluble subgroup.

As a consequence, any finite group with nilpotent commutator subgroup contains a largest supersoluble normal subgroup.
It is possible to prove that if G is a finite group G with nilpotent commutator subgroup admitting a triple factorization

$$G = AB = AC = BC,$$

where A,B and C are supersoluble subgroups, then G itself is supersoluble.

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Recall that a group G has *finite abelian section rank* if it has no infinite elementary abelian p-sections for any prime p.

The situation is much more complicated for infinite groups, as Y.P. Sysak constructed groups which are not locally supersoluble but have a triple factorization by abelian subgroups.

Recall that a group G has *finite abelian section rank* if it has no infinite elementary abelian p-sections for any prime p.

Theorem (S. Franciosi and F. de Giovanni, 1997)

Let the group

$$G = AB = AC = BC$$

be the product of three locally supersoluble subgroups A, B and C. If C has finite abelian section rank and the commutator subgroup G' of G is locally nilpotent, then G is locally supersoluble.

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In the example of Sysak, every non-trivial element has infinitely many conjugates.

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Let the group $G = AB$ be the product of two abelian subgroups A and B. Then every chief factor of G is centralized either by A or by B.

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Corollary

Let the group

$$G = AB = AC = BC$$

be the product of three abelian subgroups A, B, C. Then all chief factors of G are central.

Main Result

Theorem (A.A. and F. de Giovanni, 2011)

Let the FC-hypercentral group

$$G = AB = AC = BC$$

be the product of three locally supersoluble subgroups A, B, C. If the commutator subgroup G' of G is nilpotent, then G is locally supersoluble.

Definition

A group G is said to be hypercentral if it coincides with its hypercentre, or equivalently if any non-trivial homomorphic image of G has non-trivial centre.

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Lemma

Let G be an FC-hypercentral group whose chief factors are central. Then G is hypercentral.

Theorem (R. Baer)

If G is a finite group with nilpotent commutator subgroup, then every collection of supersoluble normal subgroups of G generates a supersoluble subgroup.

Lemma (R. Baer)

Let G be a group with locally nilpotent commutator subgroup, and let H and K be locally supersoluble normal subgroups of G. Then the subgroup HK is locally supersoluble.

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Let N be a normal subgroup of a group G and suppose that N and G/N' are nilpotent. Then G is nilpotent.

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Let N be a normal subgroup of a group G and suppose that N and G/N' are nilpotent. Then G is nilpotent.

Similarly, let N be a nilpotent normal subgroup of a group G and suppose that G/N' is locally supersoluble. Then G is locally supersoluble.

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Any group G has a largest hypercyclically embedded normal subgroup H, which is of course characteristic, and G is locally supersoluble if and only if G/H has the same property.

Main Result

Theorem (A.A. and F. de Giovanni, 2011)

Let the FC-hypercentral group

$$G = AB = AC = BC$$

be the product of three locally supersoluble subgroups A, B, C. If the commutator subgroup G' of G is nilpotent, then G is locally supersoluble.

Lemma (A.A. and F. de Giovanni, 2011)

Let the FC-hypercentral group

$$G = AB = AK = BK$$

be the product of two abelian subgroups A and B and a locally supersoluble normal subgroup K. Then G is locally supersoluble.

Lemma (A.A. and F. de Giovanni, 2011)

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Definition

A group G is said to be hypercyclic if any non-trivial homomorphic image of G contains a cyclic non-trivial normal subgroup.

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Definition

A group G is said to be hypercyclic if any non-trivial homomorphic image of G contains a cyclic non-trivial normal subgroup.

Recall a result of D.H. McLain: every finitely generated FC-hypercentral group G contains a nilpotent subgroup of finite index.

Sketch of the proof

- It is enough to prove that G/G'' is locally supersoluble.

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- Let K be the largest hypercyclically embedded normal subgroup of AG'.
- Consider the factor group $\bar{G} = G/K$.
- $A \cap G'$ is hypercyclically embedded in AG'.
- Therefore $A \cap G'$ is contained in K and hence $\bar{A} \cap \bar{G}' = \{1\}$.

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- It can be supposed that G is metabelian.
- Let K be the largest hypercyclically embedded normal subgroup of AG'.
- Consider the factor group $\bar{G} = G/K$.
- $A \cap G'$ is hypercyclically embedded in AG'.
- Therefore $A \cap G'$ is contained in K and hence $\bar{A} \cap \bar{G}' = \{1\}$.
- The subgroup $\bar{A}$ is abelian.

Sketch of the proof

- Let $\bar{L}$ be the largest hypercyclically embedded normal subgroup of $\bar{B}\bar{G}'$.

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- Set $\tilde{G} = \hat{G}/\hat{N}$.

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- Let $\bar{L}$ be the largest hypercyclically embedded normal subgroup of $\bar{B}\bar{G}'$.
- Set $\hat{G} = \bar{G}/\bar{L}$.
- We obtain that $\hat{B}$ is abelian.
- Let $\hat{N}$ be the largest hypercyclically embedded normal subgroup of $\hat{C}\hat{G}'$
- Set $\tilde{G} = \hat{G}/\hat{N}$.
- We obtain that $\tilde{C}$ is abelian.

Sketch of the proof

- Therefore

$$\tilde{G} = \tilde{A}\tilde{B} = \tilde{A}\tilde{C} = \tilde{B}\tilde{C}$$

is the product of its abelian subgroups $\tilde{A}$,$\tilde{B}$ and $\tilde{C}$.

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- All chief factors of $\tilde{G}$ are central.

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- All chief factors of $\tilde{G}$ are central.
- $\tilde{G} = \hat{G}/\hat{N}$ is hypercentral.

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- $\hat{C}\hat{G}'$ is locally supersoluble.

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- $\tilde{G} = \hat{G}/\hat{N}$ is hypercentral.
- $\hat{C}\hat{G}'$ is locally supersoluble.
- $\hat{G} = \hat{A}\hat{B} = \hat{A}(\hat{C}\hat{G}') = \hat{B}(\hat{C}\hat{G}')$ has a triple factorization by the abelian subgroups $\hat{A}$ and $\hat{B}$ and by the locally supersoluble normal subgroup $\hat{C}\hat{G}'$.

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- $\hat{G} = \bar{G}/\bar{L}$ is locally supersoluble.
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- Consider the metabelian group $\bar{G} = (\bar{B}\bar{G}')(\bar{C}\bar{G}')$.
- $\bar{G} = G/K$ is locally supersoluble.

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