Modules with maximal growth over free group algebras

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This talk is dedicated to the memory of Narain Gupta

The contents of the present talk is based on recent joint papers with **Alexander Olshanskii**:

1. "SCHREIER REWRITING BEYOND THE CLASSICAL SETTING", *Science in China Series A: Mathematics*, **52** (2009), 231–243;
2. "ACTIONS OF MAXIMAL GROWTH", *Proceedings London Math. Soc. (3)*, **101** (2010), 27 - 72.

A number of results of Professor Narain Gupta have been devoted to the Free Group Rings. Some of them are summarized in a book

Narain D. Gupta, FREE GROUP RINGS, Contemporary Mathematics **66**. American Mathematical Society, Providence, RI, 1987. xii+129 pp.

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Free group rings turn out to be "relations", in the sense of P.M.Cohn, to the free associative rings as implied by the title of of his monograph "FREE RINGS AND THEIR RELATIONS", 2nd edition, Academic Press, London, 1985.

The main notion in this latter book is a *free ideal ring (fir)* that is a ring in which all one sided ideals are free modules. A free group ring (algebra) and a free associative ring (algebra) are examples of firs.

Some of the results about modules over free group algebras and free associative algebra are in line with similar results in the Group Theory, some look even stronger. Below are some results of P.M.Cohn, G. Bergman and J. Lewin:

1. Submodules of free modules are free of unique rank.
2. Full analogue of Schreier's Formula for the rank of the submodule of finite codimension.
3. Finitely related module always has a free submodule of finite codimension.
4. A module given by p generators and q relations, with $p - q > 0$ always has a nonzero free submodule of finite codimension.

Let $\mathcal{F}_r$ (resp., $\mathcal{A}_r$) be the group algebra of a free group F_r (or free associative algebra) of rank $r > 1$, over a field Φ. By way of generalizing free modules over free associative and free group algebras one can consider *large modules* and *modules of maximal growth*.

Definition

A right R- module M over a field Φ is called *large* if it has a submodule N of finite codimension over Φ which can be mapped onto R_R.

As seen from above, any module with p generators and q relators over $R = \mathcal{F}_r$ or $R = \mathcal{A}_r$ is large as soon as $p - q > 0$.

Even more general class arises if one considers the *growth* of modules. Let us fix a symmetric basis $Y = \{x_1^{\pm 1}, \ldots, x_r^{\pm 1}\}$ in the free group F_r of rank $r > 1$. Suppose a right, say $\mathcal{F}_r$-module M is generated by a finite set A of generators. Applying to A the elements of Y, we obtain an increasing chain of subspaces

$$A_0 \subset A_1 \subset A_2 \subset \ldots$$

such that

$$A_0 = \Phi A,\ A_1 = A_0 + A_0 \cdot Y,\ A_2 = A_1 + A_1 \cdot Y \ldots$$

We set $g(n) = \dim A_n$. This is called the *growth function* of M with respect to A. If we replace A by another finite generating set B and obtain the growth function $h(n)$ then there is a constant C such that $g(n) \leq h(n + C)$ and $h(n) \leq g(n + C)$.

Two functions $g(n)$ and $h(n)$ satisfying these conditions are called equivalent. The equivalence class of the growth functions is called the *growth*. A finitely generated module M whose growth is the same as the growth of the cyclic free module (=any finitely generated free module) is called a *module of maximal growth*. In this case of a finitely generated free module, any growth function is equivalent to $(2r-1)^n$. In the case of not necessary finite generation, a module is of maximal growth if it has a cyclic submodule of maximal growth.

All large modules have maximal growth. Since all infinite - dimensional finitely presented modules have a free submodule of finite codimension, it follows that they have maximal growth.

Now let Δ be the augmentation ideal of $\mathcal{F}_r$. A right module M over $\mathcal{F}_r$ is called *nil* (resp., *periodic*) if for any $(x, u) \in M \times \Delta$ (resp., $(x, g) \in M \times F_r$) there is $n > 0$ such that $xu^n = 0$ (resp., $xg^n = x$). The main results of this talk are that nil or periodic $\mathcal{F}_r$-modules can be constructed with the maximal growth.

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In the case of nil-modules we use large modules.

Theorem

Let R is a free associative algebra or a free group algebra of rank $r > 1$ and Δ the augmentation ideal of R. Suppose an R-module M is given a presentation with p generators and q relations such that $p - q > 0$. Then M is large and a submodule L of finite codimension which can be mapped onto R can be chosen in such a way that $M\Delta^k \subset L$ for almost all natural k.

Using appropriately designed direct limits of large modules, we can prove the following.

Theorem

Any $\mathcal{F}_r$-module M presented by p generators and q relators, such that $p - q > 0$, can be mapped onto an infinite - dimensional finitely generated nil-$\mathcal{F}_r$-module of maximal growth. This latter can be mapped onto a module, which is still infinite-dimensional and nil but also residually finite-dimensional.

A construction of a periodic module of maximal growth is based on a result of the paper in Proc. London Math. Soc. 2010 dealing with the actions of free groups on the sets (F_r-sets).

In this case one can also speak about the growth of a finitely generated F_r-set S. Again, the growth of S is maximal if it contains a cyclic F_r-subset whose growth function is equivalent to $(2r-1)^n$. The cyclic F_r-sets are those of the form F_r/H, where H is a subgroup. If H is finitely generated and of infinite index, then this action is definitely of maximal growth.

The study of F_r-sets amounts to the study of so called *Schreier graphs*, generalizing Cayley graphs, and involves some "surgical" procedures, which can be quite delicate. In this geometrical way the following can be proved.

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Theorem

Any finitely generated subgroup H of infinite index in the free group F_r of rank $r > 1$, is contained as a free factor in a subgroup K of infinite index in F_r such that for any natural k the right action of F_r on F_r/K is k-transitive. In particular, K is a maximal subgroup in F_r. One can choose K in such a way that the growth of the action of F_r on F_r/K is maximal.

Definition

A subgroup K of a free group F is called Burnside if $g^n \in K$ for every element $g \in F$, where $n = n(g)$ is a positive integer.

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Theorem

Any finitely generated subgroup H of infinite index in a free group F_r of rank $r > 1$, is a free factor in a Burnside subgroup K of infinite index such that for any natural k the action of F_r on F_r/K is k-transitive. One can choose K so that the growth of the action of F_r on F_r/K is maximal.

One could obtain examples of Burnside subgroups by taking H normal such that F_r/H is an infinite Burnside group. However, in this case the growth of the action of F_r on F_r/H is not of maximal. This is quite a general property called the *faithfulness* of maximal growth and applies also to modules over $\mathcal{F}_r$.

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Considering linear spans of the actions of the form F_r/K as above, one can quite smoothly arrive at our final result.

Theorem

There exists a finitely generated simple and periodic $\mathcal{F}_r$-module of maximal growth.