Cocharacters of $UT_2(E)$

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Polynomial identities

Let F be a field and $X = \{x_1, x_2, \dots\}$ a countable set of indeterminates. We denote by $F\langle X\rangle$ the free associative algebra freely generated by X and we call polynomials its elements.

Definition

Let A be a F-algebra and $f(x_1, \dots, x_n) \in F\langle X\rangle$. We say that $f(x_1, \dots, x_n)$ is a polynomial identity for A if $f(a_1, \dots, a_n) = 0$ for any $a_1, \dots, a_n \in A$. In particular, if there exists a non-trivial polynomial identity for A, we say that A is an algebra with polynomial identities or, briefly, PI-algebra.

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We consider $T(A)$, the ideal of all polynomial identities of A and the space V_n of all multilinear polynomials in n indeterminates. If the characteristic of F is 0, $T(A)$ is generated, as a T-ideal, by the subspaces $V_n \cap T(A)$.

Remark

It is more efficient to study the factor space $V_n(A) = V_n/(V_n \cap T(A))$ in fact, although the intersection $V_n \cap T(A)$ is huge as n goes to infinity, if A is a PI-algebra, $V_n(A)$ grows at most exponentially (see the famous work of Regev [11])

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Motivation 1

An effective tool for the study of $V_n(A)$ is provided by the representation theory of the symmetric group. Indeed, one can notice that $V_n(A)$ is an S_n-module, then we consider the following:

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We call n-th cocharacter of A the character of the S_n-module $V_n(A)$.

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We recall that a partition λ of the non-negative integer n ($\lambda \vdash n$) is a sequence of integers $\lambda = (\lambda_1, \ldots, \lambda_r)$ such that:

$$\lambda_1 \geq \cdots \geq \lambda_r \geq 1 \quad \text{and} \quad \lambda_1 + \cdots + \lambda_r = n$$

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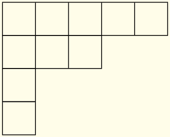
Definition

Given a partition $\lambda = (\lambda_1, \dots, \lambda_r)$, we associate to λ the skew table $[\lambda]$ having r rows and the i-th row contains λ_i squares. We call $[\lambda]$ the Young diagram of λ.

For instance, the partition $(5, 3, 1, 1)$ is associated to the following Young diagram:



It follows that for any PI-algebra A,

$$\chi_n(A) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda,$$

where the χ_λ's are the characters associated to the irreducible submodules of $V_n(A)$.

Higher commutators

Let A be a F-algebra, then for any $a, b \in A$ we denote by $[a, b]$ the element $ab - ba$.

Definition

We define higher commutators of degree $n \geq 3$ (or simply commutators) inductively, i.e.:

$$[a_1, \dots, a_n] := [[a_1, \dots, a_{n-1}], a_n].$$

The Grassmann algebra

Definition

Let us consider $F\langle X\rangle$ and its two-sided ideal I generated by the set of polynomials $\{x_ix_j + x_jx_i | i, j \geq 1\}$. We shall call infinite dimensional Grassmann algebra the factor space $E = F\langle X\rangle/I$.

We have the following result:

Theorem

The T-ideal of polynomial identities of E is generated, as a T-ideal, by the triple commutator $[x_1, x_2, x_3]$.

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Motivation 2

The explicit form of the multiplicities of $\chi_n(A)$ is known for few algebras only, among them:

- the Grassmann algebra E (Olsson and Regev [8]),
- the 2×2 matrix algebra $M_2(F)$ with entries in F (Formanek [7], Drensky [6]),
- the algebra $UT_2(F)$ of the 2×2 upper triangular matrices with entries in F (Mishchenko, Regev and Zaicev [9]),
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Cocharacters of the Grassmann algebra E

Theorem

Let E be the infinite dimensional Grassmann algebra, then for any $n \geq 1$:

$$\chi_n(E) = \sum_{k=1}^{n} \chi_{(k,1^{n-k})}.$$

The irreducibles $\chi_{(k,1^{n-k})}$ look like

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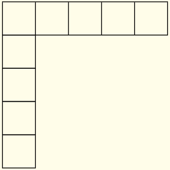

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Factorable T-ideals

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Let A be a PI-algebra. If there exist PI-algebras $A_1, A_2, \ldots, A_r$ such that $T(A) = T(A_1)\cdots T(A_r)$, we say that A has a factorable T-ideal.

In particular, if $T(A) = T(A_1)T(A_2)$, we have a nice formula for $\chi_n(A)$ due to Berele and Regev:

$$\chi_n(A) = \chi_n(A_1) + \chi_n(A_2) + \chi_{(1)} \widehat{\otimes} \sum_{k=0}^{n-1} \chi_k(A_1) \otimes \chi_{n-1-k}(A_2) +$$

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The algebra $UT_2(E)$

We consider the algebra of upper triangular matrices

$$UT_2(E) := \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in E \right\}$$

with entries in the Grassmann algebra E. It is well known that $UT_2(E)$ has a factorable T-ideal and, in particular,

$$T(UT_2(E)) = T(E)T(E).$$

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There is a strict connection between the multiplicities of cocharacters and those of proper cocharacters:

Theorem (Drensky [5])

Let A be a PI-algebra and $\chi_n(A) = \sum_{\lambda \vdash n} m_\lambda(A)\chi_\lambda$ its n-th cocharacter. Let $\xi_p(A) = \sum_{\nu \vdash p} k_\nu(A)\xi_\nu$ its p-th proper cocharacter, then

$$m_\lambda(A) = \sum_{\nu \in S} k_\nu(A),$$

where
$S = \{\nu = (\nu_1, \ldots, \nu_n) \mid \lambda_1 \geq \nu_1 \geq \lambda_2 \geq \nu_2 \geq \cdots \geq \lambda_n \geq \nu_n\}$.

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The previous theorem gives also an idea of how to recollect the cocharacters from the proper ones. The intertwining partitions ν as in the theorem of Drensky are exactly the only admissible partitions participating in the multiplication by the Young rule.

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$$\sum k_\nu \underbrace{\yng(4,2,1,1)}_{\nu} \otimes \sum \yng(4) = \sum m_\lambda \underbrace{\yng(5,3,2,2,1)}_{\lambda}.$$

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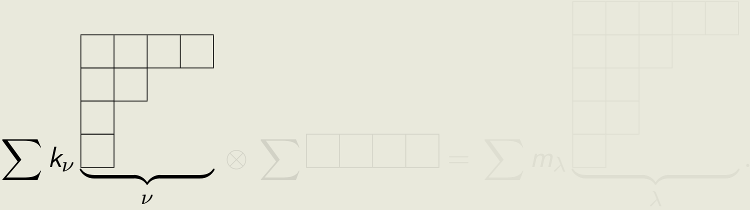

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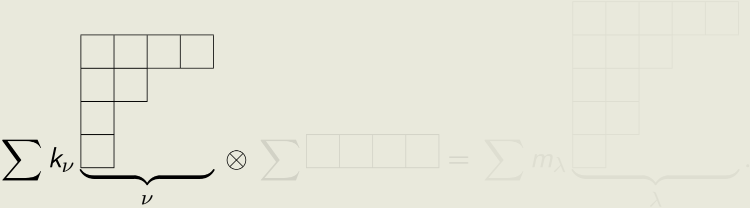

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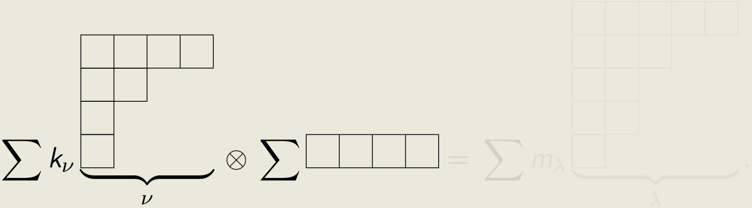

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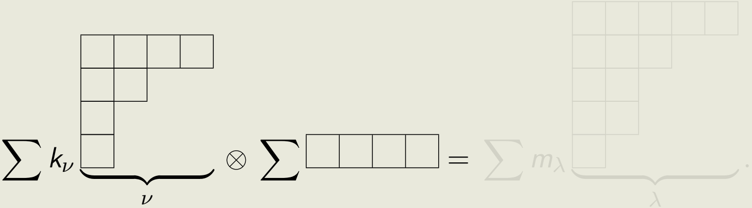

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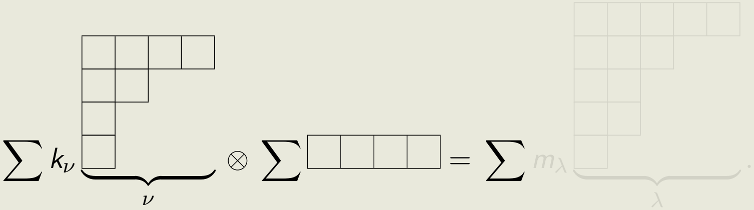

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The proper cocharacters of the Grassmann algebra

Theorem

Let E be the infinite dimensional Grassmann algebra, then for any $n \geq 1$:

$$\xi_n(E) = \begin{cases} \chi_{\emptyset} & \text{if } n \text{ is odd;} \\ \chi_{(1^n)} & \text{if } n \text{ is even.} \end{cases}$$

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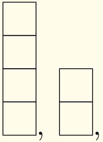

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Using the tool of Hilbert series, we derive a formula similar to that of Berele and Regev for proper cocharacters of factorable T-ideals:

Theorem ([3])

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$$= \xi_n(A_1)+\xi_n(A_2)+\sum_{l=0}^{n-1}\xi_{(l)}\widehat{\otimes}\xi_{(1)}\widehat{\otimes}\sum_{k=0}^{n-l-1}\xi_k(A_1)\otimes\xi_{n-l-1-k}(A_2)+$$

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Observation

Tensoring with strips is more efficient than tensoring with hooks. In fact, we may use the well known "Young rule" in order to compute the proper cocharacters of $UT_2(E)$. Now in the light of the work of Drensky we just have to use once more the Young rule in order to recollect the cocharacters of $UT_2(E)$.

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Theorem (see C. [3])

$\chi_n(UT_2(E)) = \sum m_\lambda \chi_\lambda$, *where*

$$\lambda = (k_1, k_2, 2^m, 1^l) \text{ or } \lambda = (k_1, k_2, 3, 2^m, 1^l).$$

If $\lambda = (k_1, k_2, 2^m, 1^l)$, *then*

$$m_\lambda = 12(k_1 - k_2 + 1)(l + 1) \quad \text{if } k_1 \geq k_2 \geq 3, \ m \geq 1$$
$$m_\lambda = 4(k_1 - k_2 + 1)(2l + 1) \quad \text{if } k_1 \geq k_2 \geq 3, \ m = 0$$
$$m_\lambda = 8(k_1 - 2)(l + 1) + 4(l + 1) \quad \text{if } k_1 \geq k_2 = 2, \ m \geq 1$$
$$m_\lambda = 3(k_1 - 2)(2l + 1) + 3l + 2 \quad \text{if } k_1 \geq k_2 = 2, \ m = 0$$
$$m_\lambda = (k_1 - 2)(2l - 1) + l + 1 \quad \text{if } k_1 \geq 2, \ k_2 = 0, \ m = 0, l \geq 1$$
$$m_\lambda = 1 \quad \text{if } \lambda = (1^l) \text{ or } \lambda = (k).$$

If $\lambda = (k_1, k_2, 2^m, 1^l)$, *then*

$$m_\lambda = 4(k_1 - k_2 + 1)(l + 1) \ \text{if } k_2 \geq 3, \ m \geq 1.$$

PI improvement

- In [1] Boumova and Drensky gave an easy algorithm which calculates the generating function of the cocharacter sequence of the T-ideal of $UT_n(F)$.
- In [4] the author and Drensky generalize the previous algorithm to the case of upper triangular matrices $UT_n(E)$.They tried to calculate the generating function of the cocharacter sequence of the T-ideal of $UT_n(E)$.
- The new tool is the double set of indeterminates in order to control the "vertical strip" part of cocharacters and the "horizontal strip".

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- The explicit knowledge of the cocharacters of $UT_2(E)$ may be used in order to build up something similar to "Young rule" for "hooks" instead of "strips".

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Acknowledgments

I want to thank Professor O. M. Di Vincenzo for introducing me this interesting problem.

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