Groups with many inert subgroups

dedicated to the memory of Jim Wiegold

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joint **work-in-progress** with
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1. CF:=Core-Finite groups.

J.T. Buckley, J.C. Lennox, B.H. Neumann, H. Smith and J. Wiegold showed

Let G be a group whose all periodic quotients are locally finite (this happens if G is hyper locally nilpotent-or-finite). Then

(CF) $\forall H \leq G \quad |H/H_G| < \infty$ *(core-finite, G is CF)*

implies that G is abelian-by-finite, that is

(AF) $\exists A \lhd G \; : \; A$ *is abelian and G/A is finite.*

Moreover, if G is ***periodic****, it is even BCF, that is*

(BCF) $\exists n \, \forall H \leq G \quad |H/H_G| \leq n$

Conversely

$\exists A \lhd G \; : \; |G/A| < \infty \; \wedge \; (\forall H \leq A \quad |H/H_G| < \infty) \; \Rightarrow \; G$ is CF

If any subgroup of A is normal in G indeed, say G **elementary CF**.

2. groups of automorphisms.

Recall B.H.Neumann's celebrated theorem:
(FA) $\forall H \leq G \quad |H^G : H| < \infty \;\Leftrightarrow\; |G'| < \infty$ (finite-by-abelian).

Let Γ be a group of automorphisms of an abelian group A and
(AP) $\forall H \leq A \quad |H/H_\Gamma| < \infty;$ *(almost-power)*
(BP) $\forall H \leq A \quad |H^\Gamma/H| < \infty;$
If Γ is ***finitely generated****, (AP) and (BP) are both equivalent to:*
(CP) $\forall H \leq A\ \exists N = N^\Gamma \leq A \;:\; |NH/(H \cap N)| < \infty. \quad (H \sim N \lhd G)$

(AP) was considered by S.Franciosi, F.de Giovanni and M.L.Newell,
(BP) was considered by C.Casolo;

Note that for arbitrary groups the picture is more complicated:

There exist ***elementary abelian p-groups*** *A and $\Gamma \leq \mathrm{Aut} A$ which are:*
(a) AP, not BP; (b) BP, not AP; (c) CP, neither AP nor BP.

3. Description of locally finite CF-groups

By the above quoted work, it follows easyly that G if BCF.

A locally finite group G is CF iff it has a normal subgroup
$A = C \times (D \times E)$ *such that*
- A is abelian and G/A is finite ,
- $\pi(C) \cap \pi(DE) = \emptyset$ (and $\pi(D) = \pi(E)$ is finite),
- D is divisibile with finite total rank r,
- E ha finite exponent e,
- G acts on C, D, E by means of power automorphisms.

Under these circumstances $\forall H \leq G \; |H/H_G| \leq e^r \cdot |G/A|$ and $G/C_G(A)$ is abelian.

4. (non-periodic) BCF-groups

The group $(C_\infty \times C_{p^\infty}) \rtimes \langle(+1, -1)\rangle$ *is CF, but not BCF.*

On the other hand, a (non-periodic) elementary CF-group is trivially BCF.

Among abelian-by-finite CF-groups, those which are BCF are easily detected.

An abelian-by-finite CF-group G is BCF iff
there is a normal abelian subgroup B such that:
1) G acts on B by means of power automorphism,
2) G/B has finite exponent (and is therefore elementary CF).

Recall that if G is non-periodic the action G on B is the power ± 1, that is either the identity or the inversion map.

A more complete statement:

An abelian-by-finite group G is CF iff either:
- it is elementary CF or
- there is a normal abelian finite index subgroup A and a G-series

$$1 \leq V \leq A$$

1) G acts (AP) on the periodic group A/V, that is G/V is CF,
2) V is finitely generated free-abelian and G is power ± 1 on V.

Moreover G is BCF iff (1), (2) hold and there is $B \leq A$ such that

3) G/B has finite exponent and G acts on B as power ± 1.
The subgroup A may be choosen such that $G/C_G(A)$ is supersolvable and has derived lenght at most 3.

Definition

- *Subgroups H, K are told commensurable iff the index of $H \cap K$ in both H and K is finite.*

- *Write $H \,\mathrm{cn}\, G$ iff H is commensurable with a normal subgroup of G, that is $H \,\mathrm{cn}\, G \quad :\Leftrightarrow \quad \exists N \lhd G : |HN/(H \cap N)| < \infty$*

- *Call CN groups in which every subgroup is cn .*

- *CF-groups are CN*
- *finite-by-CN groups are CN.*

QUESTION: When is a CN-group finite-by-abelian-by-finite?
whence finite-by-CF.

when dealing with locally finite groups we reduce to consideration of hyperabelian-by-finite countable p-groups

A subgroup is told inert if it is commensurable with all conjugates of its.

If every subgroup of G is inert, G is told totally inert (TIN).

Clearly, CN-groups are TIN-groups.

Totally inert groups have received attention of many authors.
V.V. Belayev, M. Kuzucuoğlu and E. Seckin (1999) showed that **there are no simple locally finite infinite TIN-groups**,
M.R. Dixon, M.R. Evans, A. Tortora (2009) extended this result to **locally graded** groups,

D.Robinson (2006) treated the soluble case.

8. For S_1F-groups, CN implies finite-by-abelian-by-finite

D.Robinson has classified S_1F-groups which are TIN.

Recall that an S_1F-group is a a finite extension of a soluble group with finite abelian total rank (FATR) or -equivalently- with a finite abelian series whose factors are direct sum of finitely many copies of either Prüfer groups or subgroups of the rationals.

Using that classification we have:

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If an S_1F-group G is CN, then it is finite-by-abelian-by-finite.

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If the above G is periodic it is Chernikov, thus elementary and

$(BCN) \quad \exists n \, \forall H \leq G \, \exists N \lhd G \; : |HN/(H \cap N)| \leq n$

9. When is cn transitive?

Recall that we write $H \,\mathrm{cn}\, G$ when H is commensurable with a normal subgroup of G, that is:
$H \,\mathrm{cn}\, G \quad :\Leftrightarrow \quad \exists N \lhd G : |HN/(H \cap N)| < \infty.$

For a group G the following conditions are equivalent:
i) $\forall H \leq G, \quad H \,\mathrm{sn}\, G \Rightarrow H \,\mathrm{cn}\, G,$ *that is* sn *implies* cn
ii) $\forall H \leq K \leq G,\ H \,\mathrm{cn}\, K$ *and* $K \,\mathrm{cn}\, G \Rightarrow H \,\mathrm{cn}\, G$ cn *is transitive.*

10. $\tilde{T}$-groups whose normal torsion subgroups are trivial

Let G hyper-(abelian or finite) group
without non-trivial normal torsion subgroups.
*0) D.Robinson: if **all** subgroups are **inert**, G is abelian or dihedral*

1) all subnormal subgroups of G are inert iff
G is either abelian or dihedral or $G = A \rtimes K$ where:
- A abelian torsion free with finite rank and
- K acts faithfully on A by means of rational power automorphisms.

Recall that the above A embeds in $\mathbb{Q} \oplus \cdots \oplus \mathbb{Q}$.
Then $x \mapsto \frac{m}{n}x$ with $\frac{m}{n} \in \mathbb{Q}$ is called rational power automorphisms.

Let G as above. Then:
2) All subnormal subgroups of G are cn iff
G is either abelian or dihedral.

11. finitely generated hyper AF-groups which are TIN

For a hyper-(abelian or finite) finitely generated group G the following are equivalent:

1) all subnormal subgroups are inert;

2) G has a subgroup with finite index $G_0 = A \rtimes K$ such that:

- A is isomorphic to a subgroup of $\mathbb{Q}_\pi \oplus \cdots \oplus \mathbb{Q}_\pi$ with finite rank and π is a finite set of primes.

- G acts by rational power automorphisms on A,

- K acts faithfully on A (and K may be taken finitely generated free abelian)

- G acts by power automorphisms ± 1 on $G_0/A \simeq K$,

5) G has a finite normal subgroup F such that $G/F = \bar{A} \rtimes \bar{K}$ has the same shape as G_0 above.

In particular G'' is finite (and G is S_1F).

Let G be an hyper-(abelian or finite) finitely generated group.
Then, in G: cn *is transitive* *iff* *all subgroups are* cn.

TO BE CONTINUED... :-)

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