Groups whose subgroups of infinite rank are permutable

Martyn R. Dixon[1]

[1]Department of Mathematics
University of Alabama

Ischia Group Theory Conference, March 2012

Preliminaries

Joint work with Zekeriya Yalcin Karatas

Preliminaries

Joint work with Zekeriya Yalcin Karatas

Question

What can we say about a group all of whose proper subgroups have some property $\mathcal{P}$?

Preliminaries

Joint work with Zekeriya Yalcin Karatas

Question

What can we say about a group all of whose proper subgroups have some property $\mathcal{P}$?

Properties $\mathcal{P}$ of interest in this talk include: permutability, finite rank, and related properties. But it is a question with a rich history.

Some History

- Dedekind groups: Groups with all subgroups normal: Precisely those groups which are either abelian or $Q \times E \times O$.

Some History

- Dedekind groups: Groups with all subgroups normal: Precisely those groups which are either abelian or $Q \times E \times O$.
- Groups with all proper subgroups abelian: non-abelian groups with all proper subgroups abelian can be quite complicated-Tarski monsters.

Some History

- Dedekind groups: Groups with all subgroups normal: Precisely those groups which are either abelian or $Q \times E \times O$.
- Groups with all proper subgroups abelian: non-abelian groups with all proper subgroups abelian can be quite complicated-Tarski monsters.
- Groups G with all proper subgroups nilpotent. When G is finite then G is soluble (Schmidt).

Some History

- Dedekind groups: Groups with all subgroups normal: Precisely those groups which are either abelian or $Q \times E \times O$.
- Groups with all proper subgroups abelian: non-abelian groups with all proper subgroups abelian can be quite complicated-Tarski monsters.
- Groups G with all proper subgroups nilpotent. When G is finite then G is soluble (Schmidt). When G is infinite and locally graded then G is also soluble (Asar).

Some History

- Dedekind groups: Groups with all subgroups normal: Precisely those groups which are either abelian or $Q \times E \times O$.
- Groups with all proper subgroups abelian: non-abelian groups with all proper subgroups abelian can be quite complicated-Tarski monsters.
- Groups G with all proper subgroups nilpotent. When G is finite then G is soluble (Schmidt). When G is infinite and locally graded then G is also soluble (Asar). G is locally graded when every nontrivial finitely generated subgroup has a nontrivial finite image.

Permutable subgroups

A subgroup H of a group G is permutable if $HK = KH$ for all subgroups K of G.

Permutable subgroups

A subgroup H of a group G is permutable if $HK = KH$ for all subgroups K of G.

- (Stonehewer, 1972) Permutable subgroups are ascendant; in fact in finitely generated groups permutable subgroups are subnormal. Thus groups with all subgroups permutable are Gruenberg groups.

Permutable subgroups

A subgroup H of a group G is permutable if $HK = KH$ for all subgroups K of G.

- (Stonehewer, 1972) Permutable subgroups are ascendant; in fact in finitely generated groups permutable subgroups are subnormal. Thus groups with all subgroups permutable are Gruenberg groups.
- (Iwasawa, 1943) Classified groups with all subgroups permutable.

Permutable subgroups

A subgroup H of a group G is permutable if $HK = KH$ for all subgroups K of G.

- (Stonehewer, 1972) Permutable subgroups are ascendant; in fact in finitely generated groups permutable subgroups are subnormal. Thus groups with all subgroups permutable are Gruenberg groups.
- (Iwasawa, 1943) Classified groups with all subgroups permutable.
- If G is a nonabelian group with elements of infinite order and all subgroups permutable then $T(G)$ is abelian and $G/T(G)$ is torsionfree abelian of rank 1.

Permutable subgroups

A subgroup H of a group G is permutable if $HK = KH$ for all subgroups K of G.

- (Stonehewer, 1972) Permutable subgroups are ascendant; in fact in finitely generated groups permutable subgroups are subnormal. Thus groups with all subgroups permutable are Gruenberg groups.
- (Iwasawa, 1943) Classified groups with all subgroups permutable.
- If G is a nonabelian group with elements of infinite order and all subgroups permutable then $T(G)$ is abelian and $G/T(G)$ is torsionfree abelian of rank 1. Rather more precise information can be obtained.

Permutable subgroups continued

- Periodic groups with all subgroups permutable are therefore locally finite; they have known structure.

Permutable subgroups continued

- Periodic groups with all subgroups permutable are therefore locally finite; they have known structure.
- (Stonehewer, 1972) A simple group never contains a proper, nontrivial, permutable subgroup.

Finite Rank

- G has finite rank, r, if every finitely generated subgroup of G is at most r-generator and r is the least natural number with this property.

Finite Rank

- G has finite rank, r, if every finitely generated subgroup of G is at most r-generator and r is the least natural number with this property.
- What can be said about locally graded groups of finite rank?

Finite Rank

- G has finite rank, r, if every finitely generated subgroup of G is at most r-generator and r is the least natural number with this property.
- What can be said about locally graded groups of finite rank?
- (N. S. Černikov, 1990) Every $\mathfrak{X}$-group of finite rank is almost locally soluble. Here $\mathfrak{X}$ is a very large class of locally graded groups This generalizes well-known theorems of Shunkov (1971) and Lubotzky-Mann (1989).

Finite Rank

- G has finite rank, r, if every finitely generated subgroup of G is at most r-generator and r is the least natural number with this property.
- What can be said about locally graded groups of finite rank?
- (N. S. Černikov, 1990) Every $\mathfrak{X}$-group of finite rank is almost locally soluble. Here $\mathfrak{X}$ is a very large class of locally graded groups This generalizes well-known theorems of Shunkov (1971) and Lubotzky-Mann (1989).
- It is unknown if $\mathfrak{X}$ is the class of all locally graded groups.

Motivation

- (Evans-Kim, 2004) Suppose that G is an $\mathfrak{X}$-group with all infinite rank subgroups subnormal of defect at most d. If G has infinite rank then G is nilpotent of class dependent upon a function of d.

Motivation

- (Evans-Kim, 2004) Suppose that G is an $\mathfrak{X}$-group with all infinite rank subgroups subnormal of defect at most d. If G has infinite rank then G is nilpotent of class dependent upon a function of d.
  This generalizes a well-known theorem of Roseblade (1965).

Motivation

- (Evans-Kim, 2004) Suppose that G is an $\mathfrak{X}$-group with all infinite rank subgroups subnormal of defect at most d. If G has infinite rank then G is nilpotent of class dependent upon a function of d.
  This generalizes a well-known theorem of Roseblade (1965).
- (Kurdachenko-Smith, 2004) If G is a locally (soluble-by-finite) group of infinite rank, all of whose infinite rank subgroups are subnormal, then G is a soluble Baer group.

Motivation

- (Evans-Kim, 2004) Suppose that G is an $\mathfrak{X}$-group with all infinite rank subgroups subnormal of defect at most d. If G has infinite rank then G is nilpotent of class dependent upon a function of d.
  This generalizes a well-known theorem of Roseblade (1965).
- (Kurdachenko-Smith, 2004) If G is a locally (soluble-by-finite) group of infinite rank, all of whose infinite rank subgroups are subnormal, then G is a soluble Baer group.
- What can be said concerning groups all of whose infinite rank subgroups are pemutable?

Results

Theorem

Let G be an $\mathfrak{X}$-group of infinite rank in which every subgroup of infinite rank is permutable. Then every subgroup of G is permutable.

Results

Theorem

Let G be an $\mathfrak{X}$-group of infinite rank in which every subgroup of infinite rank is permutable. Then every subgroup of G is permutable.

Thus the structure of $\mathfrak{X}$-groups of infinite rank in which every subgroups of infinite rank is permutable is known.

Useful Background

- (De Falco, De Giovanni, Musella, Schmidt 2003) If G is a group, $H \leq G$ such that all subgroups containing H are permutable. If there exists $g \in G$ such that $g^n \notin H$ for all n then $H \lhd G$.

Useful Background

- (De Falco, De Giovanni, Musella, Schmidt 2003) If G is a group, $H \leq G$ such that all subgroups containing H are permutable. If there exists $g \in G$ such that $g^n \notin H$ for all n then $H \lhd G$.
- Use this to deduce the following. Let G be a group of infinite rank, all subgroups of infinite rank permutable. If G has a subgroup of type $A_1 \times A_2 \times \ldots \cong \mathbb{Z} \times \mathbb{Z} \times \ldots$ then G is abelian.

Useful Background

- (De Falco, De Giovanni, Musella, Schmidt 2003) If G is a group, $H \leq G$ such that all subgroups containing H are permutable. If there exists $g \in G$ such that $g^n \notin H$ for all n then $H \triangleleft G$.
- Use this to deduce the following. Let G be a group of infinite rank, all subgroups of infinite rank permutable. If G has a subgroup of type $A_1 \times A_2 \times \ldots \cong \mathbb{Z} \times \mathbb{Z} \times \ldots$ then G is abelian.

Let $A_i = \langle a_i \rangle$. Let $C = A_1 \times A_3 \times \ldots$, $D = A_2 \times A_4 \times \ldots$. Clearly $a_j^k \notin C$ for all even j, and every subgroup containing C is permutable, so $C \triangleleft G$. Likewise $D \triangleleft G$. G/C has all subgroups permutable so must be abelian; likewise G/D abelian so G is abelian.

More Useful Background

- (special case of Baer-Heineken Theorem) A radical group of infinite rank contains an abelian subgroup of infinite rank.

More Useful Background

- (special case of Baer-Heineken Theorem) A radical group of infinite rank contains an abelian subgroup of infinite rank.
- Let G be a locally nilpotent group of infinite rank in which every subgroup of infinite rank is permutable. Then any two subgroups of $T(G)$ permute.

More Useful Background

- (special case of Baer-Heineken Theorem) A radical group of infinite rank contains an abelian subgroup of infinite rank.
- Let G be a locally nilpotent group of infinite rank in which every subgroup of infinite rank is permutable. Then any two subgroups of $T(G)$ permute.
  G nonabelian implies G contains a subgroup $B = B_1 \times B_2 \times B_3 \times \dots$ where, for all i, $B_i \cong \mathbb{Z}_{p_i^{n_i}}$ for some prime p_i and for some positive integer n_i. Let $A_i = B_i \times B_{i+1} \times B_{i+2} \times \dots$. Let $g, h \in T(G)$. Then $\langle g \rangle \langle h \rangle = \cap_{i \geq 1} A_i \langle g \rangle \langle h \rangle$, since $\cap_{i \geq 1} A_i = 1$.

More Useful Background

Lemma

Let G be a periodic locally nilpotent group of infinite rank in which every subgroup of infinite rank is permutable. Then any two subgroups of G permute. Furthermore, G has a proper normal subgroup of infinite rank.

Proofs

- Simple $\mathfrak{X}$-groups with all proper subgroups finite rank are finite.

Proofs

- Simple $\mathfrak{X}$-groups with all proper subgroups finite rank are finite.

Lemma

Let G be an $\mathfrak{X}$-group of infinite rank in which every subgroup of infinite rank is permutable. Then G has a proper normal subgroup of infinite rank.

Proofs

- Simple $\mathfrak{X}$-groups with all proper subgroups finite rank are finite.

Lemma

Let G be an $\mathfrak{X}$-group of infinite rank in which every subgroup of infinite rank is permutable. Then G has a proper normal subgroup of infinite rank.

Suppose all proper normals have finite rank. Let J be the product of the proper normals. If $J \neq G$ it has finite rank and G/J is simple. By Stonehewer's result all proper subgroups of G/J have finite rank so G/J is finite so G has finite rank.

Proofs

Thus G is the product of its proper normal subgroups and it is easy to see that G is a radical group and previous remarks imply we may assume that G is not locally nilpotent.

Proofs

Thus G is the product of its proper normal subgroups and it is easy to see that G is a radical group and previous remarks imply we may assume that G is not locally nilpotent.
Thus $HP(G)$ has finite rank and so G has a normal subgroup M such that M' is hypercentral and $|G : M|$ is finite.

Proofs

Theorem

Let G be an $\mathfrak{X}$-group of infinite rank in which every subgroup of infinite rank is permutable. Then G is soluble.

Proofs

Theorem

Let G be an $\mathfrak{X}$-group of infinite rank in which every subgroup of infinite rank is permutable. Then G is soluble.

- G contains a proper normal of infinite rank, N. G/N is metabelian so $G'' \leq N$. If G'' has infinite rank then it has a proper normal M of infinite rank so $G^{(4)} \leq M$ so in any case $K = G^{(4)}$ is finite rank.

Proofs

Theorem

Let G be an $\mathfrak{X}$-group of infinite rank in which every subgroup of infinite rank is permutable. Then G is soluble.

- G contains a proper normal of infinite rank, N. G/N is metabelian so $G'' \leq N$. If G'' has infinite rank then it has a proper normal M of infinite rank so $G^{(4)} \leq M$ so in any case $K = G^{(4)}$ is finite rank.
- K is locally soluble of finite rank. Structure of locally soluble groups of finite rank implies that K is soluble.

Proofs

Theorem

Let G be an $\mathfrak{X}$-group of infinite rank in which every subgroup of infinite rank is permutable. Then G is a Gruenberg group and hence is locally nilpotent.

Proofs

Theorem

Let G be an $\mathfrak{X}$-group of infinite rank in which every subgroup of infinite rank is permutable. Then G is a Gruenberg group and hence is locally nilpotent.

Let $A = A_1 \times A_2 \times A_3 \times \dots$, $A_i \cong \mathbb{Z}_{p_i^{n_i}}$, p_i prime. Write $A = B \times C$; B, C infinite rank.

Proofs

Theorem

Let G be an $\mathfrak{X}$-group of infinite rank in which every subgroup of infinite rank is permutable. Then G is a Gruenberg group and hence is locally nilpotent.

Let $A = A_1 \times A_2 \times A_3 \times \ldots$, $A_i \cong \mathbb{Z}_{p_i^{n_i}}$, p_i prime. Write $A = B \times C$; B, C infinite rank.
If $g \in G$ has infinite order then $g^n \notin A$ for all $n \neq 0$. $\langle g \rangle B$ and $\langle g \rangle C$ are permutable, hence ascendant in G.

Proofs

Theorem

Let G be an $\mathfrak{X}$-group of infinite rank in which every subgroup of infinite rank is permutable. Then G is a Gruenberg group and hence is locally nilpotent.

Let $A = A_1 \times A_2 \times A_3 \times \ldots$, $A_i \cong \mathbb{Z}_{p_i^{n_i}}$, p_i prime. Write $A = B \times C$; B, C infinite rank.
If $g \in G$ has infinite order then $g^n \notin A$ for all $n \neq 0$. $\langle g \rangle B$ and $\langle g \rangle C$ are permutable, hence ascendant in G.
Let $x = bg^i = cg^j \in \langle g \rangle B \cap \langle g \rangle C$. Then $c^{-1}b = g^{j-i} \in A$ so $j = i$, $b = c = 1$. So $x \in \langle g \rangle$ and $\langle g \rangle = \langle g \rangle B \cap \langle g \rangle C$.

Proofs

$|g| = k < \infty$: construct a sequence of infinite rank abelian subgroups $X_1, X_2, \ldots$ and positive integers s_i such that $X_1 \gneqq X_2 \gneqq X_3 \gneqq \ldots$, $0 < s_i < k$ and $g^{s_i} \in X_i \setminus X_{i+1}$ for all i.

Proofs

$|g| = k < \infty$: construct a sequence of infinite rank abelian subgroups $X_1, X_2, \ldots$ and positive integers s_i such that $X_1 \gneqq X_2 \gneqq X_3 \gneqq \ldots$, $0 < s_i < k$ and $g^{s_i} \in X_i \setminus X_{i+1}$ for all i.
There exist positive integers l and m such that $g^{s_l} = g^{s_m}$; $l > m$.
$g^{s_m} \in X_m \setminus X_l$ since $l > m$, but $g^{s_l} \in X_l$, a contradiction.

Proofs

Lemma

Let G be an $\mathfrak{X}$-group of infinite rank in which every subgroup of infinite rank is permutable. Let g be an element of infinite order and h be an element of finite order. Then $\langle g\rangle \langle h\rangle$ is a subgroup of G and hence $\langle g\rangle$ and $\langle h\rangle$ permute.

Proofs

Lemma

Let G be an $\mathfrak{X}$-group of infinite rank in which every subgroup of infinite rank is permutable. Let g be an element of infinite order and h be an element of finite order. Then $\langle g \rangle \langle h \rangle$ is a subgroup of G and hence $\langle g \rangle$ and $\langle h \rangle$ permute.

Lemma

Let G be an $\mathfrak{X}$-group of infinite rank in which every subgroup is permutable or of finite rank. If G has elements of infinite order, then every subgroup of $T(G)$ is normal in G.

Proofs

$x \in T(G)$, y of infinite order. Then $\langle x \rangle \langle y \rangle \leq G$ and $\langle x \rangle \langle y \rangle \cap T(G) \trianglelefteq \langle x \rangle \langle y \rangle$. Then $y \in N_G(\langle x \rangle)$ so $\langle x \rangle \triangleleft G$.

Proof of main theorem

G is locally nilpotent and we may assume not abelian.

Proof of main theorem

G is locally nilpotent and we may assume not abelian.
It is enough to prove that $\langle x \rangle \langle y \rangle = \langle y \rangle \langle x \rangle$ for all $x, y \in G$.

Proof of main theorem

G is locally nilpotent and we may assume not abelian.
It is enough to prove that $\langle x\rangle \langle y\rangle = \langle y\rangle \langle x\rangle$ for all $x, y \in G$.
We may assume that x and y have infinite order.

Proof of main theorem

G is locally nilpotent and we may assume not abelian.
It is enough to prove that $\langle x\rangle\langle y\rangle = \langle y\rangle\langle x\rangle$ for all $x, y \in G$.
We may assume that x and y have infinite order.
G has a permutable subgroup A of infinite rank such that $A = A_1 \times A_2 \times A_3 \times \ldots$ where $A_i = \langle a_i\rangle \cong \mathbb{Z}_{p_i^{n_i}}$, p_i is a prime and n_i is a positive integer for all i.

Proof of main theorem

G is locally nilpotent and we may assume not abelian.
It is enough to prove that $\langle x \rangle \langle y \rangle = \langle y \rangle \langle x \rangle$ for all $x, y \in G$.
We may assume that x and y have infinite order.
G has a permutable subgroup A of infinite rank such that $A = A_1 \times A_2 \times A_3 \times \dots$ where $A_i = \langle a_i \rangle \cong \mathbb{Z}_{p_i^{n_i}}$, p_i is a prime and n_i is a positive integer for all i.

- $T(G)$ is abelian and $G/T(G)$ is a torsion-free abelian group of rank one.

Proof of main theorem

G is locally nilpotent and we may assume not abelian.
It is enough to prove that $\langle x\rangle\langle y\rangle = \langle y\rangle\langle x\rangle$ for all $x, y \in G$.
We may assume that x and y have infinite order.
G has a permutable subgroup A of infinite rank such that $A = A_1 \times A_2 \times A_3 \times \dots$ where $A_i = \langle a_i\rangle \cong \mathbb{Z}_{p_i^{n_i}}$, p_i is a prime and n_i is a positive integer for all i.

- $T(G)$ is abelian and $G/T(G)$ is a torsion-free abelian group of rank one.

Write $A = B \times C$; B, C have infinite rank. Every subgroup of $T(G)$ is normal so B, C are normal subgroups of G. All subgroups of G/B and G/C are permutable so $T(G/B) = T(G)/B$ and $T(G/C) = T(G)/C$ are abelian. Then $T(G) \hookrightarrow T(G)/B \times T(G)/C$ implies that $T(G)$ is abelian.

Proof of main theorem

If G/B and G/C are both abelian then $G \hookrightarrow G/B \times G/C$. implies G is abelian. Thus one of G/B or G/C is nonabelian, say G/B is nonabelian. Then $G/T(G) \cong (G/B)/T(G/B)$ is torsion-free abelian of rank one.

Proof of main theorem

If G/B and G/C are both abelian then $G \hookrightarrow G/B \times G/C$. implies G is abelian. Thus one of G/B or G/C is nonabelian, say G/B is nonabelian. Then $G/T(G) \cong (G/B)/T(G/B)$ is torsion-free abelian of rank one.
$T(G)$ has infinite rank. We need to show that $\langle x\rangle\langle y\rangle T(G)$ is a group with all subgroups permutable. Since $\langle x, y\rangle T(G)/T(G)$ is finitely generated it is cyclic. Thus for the remainder of the proof we may assume that $G/T(G)$ is infinite cyclic.

Proof of main theorem

If G/B and G/C are both abelian then $G \hookrightarrow G/B \times G/C$. implies G is abelian. Thus one of G/B or G/C is nonabelian, say G/B is nonabelian. Then $G/T(G) \cong (G/B)/T(G/B)$ is torsion-free abelian of rank one.
$T(G)$ has infinite rank. We need to show that $\langle x \rangle \langle y \rangle T(G)$ is a group with all subgroups permutable. Since $\langle x, y \rangle T(G)/T(G)$ is finitely generated it is cyclic. Thus for the remainder of the proof we may assume that $G/T(G)$ is infinite cyclic.
In fact G is the semidirect product of $T(G)$ by an infinite cyclic group $\langle z \rangle$, and for every prime p there exists a p-adic unit $r(p)$ with $r(p) \equiv 1 \pmod p$ and $r(2) \equiv 1 \pmod 4$ such that $a^z = a^{r(p)}$ for all $a \in T(G)_p$, the p-component of $T(G)$. Then the main theorem follows by the structure theorem for groups with all subgroups permutable.