Groups whose subgroups of infinite rank are permutable

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Joint work with Zekeriya Yalcin Karatas

What can we say about a group all of whose proper subgroups have some property \( P \)? Properties \( P \) of interest in this talk include: permutability, finite rank, and related properties. But it is a question with a rich history.

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- Dedekind groups: Groups with all subgroups normal: Precisely those groups which are either abelian or $Q \times E \times O$. 

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- Groups $G$ with all proper subgroups nilpotent. When $G$ is finite then $G$ is soluble (Schmidt). When $G$ is infinite and locally graded then $G$ is also soluble (Asar).
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- **Dedekind groups**: Groups with all subgroups normal: Precisely those groups which are either abelian or $Q \times E \times O$.

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- Groups $G$ with all proper subgroups nilpotent. When $G$ is finite then $G$ is soluble (Schmidt). When $G$ is infinite and locally graded then $G$ is also soluble (Asar). $G$ is locally graded when every nontrivial finitely generated subgroup has a nontrivial finite image.
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Permutable subgroups

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- If $G$ is a nonabelian group with elements of infinite order and all subgroups permutable then $T(G)$ is abelian and $G/T(G)$ is torsionfree abelian of rank 1. Rather more precise information can be obtained.
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(Stonehewer, 1972) A simple group never contains a proper, nontrivial, permutable subgroup.
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Finite Rank

- \( G \) has **finite rank, \( r \),** if every finitely generated subgroup of \( G \) is at most \( r \)-generator and \( r \) is the least natural number with this property.
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Finite Rank

- $G$ has **finite rank**, $r$, if every finitely generated subgroup of $G$ is at most $r$-generator and $r$ is the least natural number with this property.

- **What can be said about locally graded groups of finite rank?**

- **(N. S. Černikov, 1990)** Every $\mathcal{X}$-group of finite rank is almost locally soluble. Here $\mathcal{X}$ is a very large class of locally graded groups. This generalizes well-known theorems of Shunkov (1971) and Lubotzky-Mann (1989).
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It is unknown if \( \mathcal{X} \) is the class of all locally graded groups.
(Evans-Kim, 2004) Suppose that $G$ is an $\mathcal{X}$-group with all infinite rank subgroups subnormal of defect at most $d$. If $G$ has infinite rank then $G$ is nilpotent of class dependent upon a function of $d$. 

(Kurdachenko-Smith, 2004) If $G$ is a locally (soluble-by-finite) group of infinite rank, all of whose infinite rank subgroups are subnormal, then $G$ is a soluble Baer group.

What can be said concerning groups all of whose infinite rank subgroups are permutable?
Motivation

(Evans-Kim, 2004) Suppose that $G$ is an $\mathfrak{x}$-group with all infinite rank subgroups subnormal of defect at most $d$. If $G$ has infinite rank then $G$ is nilpotent of class dependent upon a function of $d$. This generalizes a well-known theorem of Roseblade (1965).
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- **What can be said concerning groups all of whose infinite rank subgroups are permutable?**
Let $G$ be an $\mathfrak{X}$-group of infinite rank in which every subgroup of infinite rank is permutable. Then every subgroup of $G$ is permutable.
Let $G$ be an $\mathcal{K}$-group of infinite rank in which every subgroup of infinite rank is permutable. Then every subgroup of $G$ is permutable.

Thus the structure of $\mathcal{K}$-groups of infinite rank in which every subgroups of infinite rank is permutable is known.
(De Falco, De Giovanni, Musella, Schmidt 2003) If $G$ is a group, $H \leq G$ such that all subgroups containing $H$ are permutable. If there exists $g \in G$ such that $g^n \notin H$ for all $n$ then $H \triangleleft G$. 
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Use this to deduce the following. Let $G$ be a group of infinite rank, all subgroups of infinite rank permutable. If $G$ has a subgroup of type $A_1 \times A_2 \times \ldots \cong \mathbb{Z} \times \mathbb{Z} \times \ldots$ then $G$ is abelian.
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Let \( A_i = \langle a_i \rangle \). Let \( C = A_1 \times A_3 \times \ldots \), \( D = A_2 \times A_4 \times \ldots \). Clearly \( a_j^k \notin C \) for all even \( j \), and every subgroup containing \( C \) is permutable, so \( C \triangleleft G \). Likewise \( D \triangleleft G \). \( G/C \) has all subgroups permutable so must be abelian; likewise \( G/D \) abelian so \( G \) is abelian.
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Let $G$ be a locally nilpotent group of infinite rank in which every subgroup of infinite rank is permutable. Then any two subgroups of $T(G)$ permute.
(special case of **Baer-Heineken Theorem**) A radical group of infinite rank contains an abelian subgroup of infinite rank.

Let $G$ be a locally nilpotent group of infinite rank in which every subgroup of infinite rank is permutable. Then any two subgroups of $T(G)$ permute. $G$ nonabelian implies $G$ contains a subgroup $B = B_1 \times B_2 \times B_3 \times \ldots$ where, for all $i$, $B_i \cong \mathbb{Z}_{p_i^{n_i}}$ for some prime $p_i$ and for some positive integer $n_i$. Let $A_i = B_i \times B_{i+1} \times B_{i+2} \times \ldots$. Let $g, h \in T(G)$. Then $\langle g \rangle \langle h \rangle = \cap_{i \geq 1} A_i \langle g \rangle \langle h \rangle$, since $\cap_{i \geq 1} A_i = 1$. 
Lemma

Let $G$ be a periodic locally nilpotent group of infinite rank in which every subgroup of infinite rank is permutable. Then any two subgroups of $G$ permute. Furthermore, $G$ has a proper normal subgroup of infinite rank.
Simple $\mathcal{X}$-groups with all proper subgroups finite rank are finite.
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**Lemma**

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**Lemma**

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Suppose all proper normals have finite rank. Let $J$ be the product of the proper normals. If $J \neq G$ it has finite rank and $G/J$ is simple. By Stonehewer’s result all proper subgroups of $G/J$ have finite rank so $G/J$ is finite so $G$ has finite rank.
Thus $G$ is the product of its proper normal subgroups and it is easy to see that $G$ is a radical group and previous remarks imply we may assume that $G$ is not locally nilpotent.
Thus $G$ is the product of its proper normal subgroups and it is easy to see that $G$ is a radical group and previous remarks imply we may assume that $G$ is not locally nilpotent. Thus $HP(G)$ has finite rank and so $G$ has a normal subgroup $M$ such that $M'$ is hypercentral and $|G : M|$ is finite.
Theorem

Let $G$ be an $\infty$-group of infinite rank in which every subgroup of infinite rank is permutable. Then $G$ is soluble.
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- $G$ contains a proper normal of infinite rank, $N$. $G/N$ is metabelian so $G'' \leq N$. If $G''$ has infinite rank then it has a proper normal $M$ of infinite rank so $G^{(4)} \leq M$ so in any case $K = G^{(4)}$ is finite rank.
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1. $G$ contains a proper normal of infinite rank, $N$. $G/N$ is metabelian so $G'' \leq N$. If $G''$ has infinite rank then it has a proper normal $M$ of infinite rank so $G^{(4)} \leq M$ so in any case $K = G^{(4)}$ is finite rank.

2. $K$ is locally soluble of finite rank. Structure of locally soluble groups of finite rank implies that $K$ is soluble.
Theorem

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Let $A = A_1 \times A_2 \times A_3 \times \ldots$, $A_i \cong \mathbb{Z}_{p_i^{n_i}}$, $p_i$ prime. Write $A = B \times C$; $B, C$ infinite rank.
Proofs

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If $g \in G$ has infinite order then $g^n \notin A$ for all $n \neq 0$. $\langle g \rangle B$ and $\langle g \rangle C$ are permutable, hence ascendant in $G$. 
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Let $x = bg^i = cg^j \in \langle g \rangle B \cap \langle g \rangle C$. Then $c^{-1}b = g^{j-i} \in A$ so $j = i$, $b = c = 1$. So $x \in \langle g \rangle$ and $\langle g \rangle = \langle g \rangle B \cap \langle g \rangle C$. 

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$|g| = k < \infty$: construct a sequence of infinite rank abelian subgroups $X_1, X_2, \ldots$ and positive integers $s_i$ such that $X_1 \not\supseteq X_2 \not\supseteq X_3 \not\supseteq \ldots$, $0 < s_i < k$ and $g^{s_i} \in X_i \setminus X_{i+1}$ for all $i$. 
$|g| = k < \infty$: construct a sequence of infinite rank abelian subgroups $X_1, X_2, \ldots$ and positive integers $s_i$ such that $X_1 \nsubseteq X_2 \nsubseteq X_3 \nsubseteq \ldots$, $0 < s_i < k$ and $g^{s_i} \in X_i \setminus X_{i+1}$ for all $i$. There exist positive integers $l$ and $m$ such that $g^{s_l} = g^{s_m}$; $l > m$. $g^{s_m} \in X_m \setminus X_l$ since $l > m$, but $g^{s_l} \in X_l$, a contradiction.
Lemma

Let $G$ be an $\infty$-group of infinite rank in which every subgroup of infinite rank is permutable. Let $g$ be an element of infinite order and $h$ be an element of finite order. Then $\langle g \rangle \langle h \rangle$ is a subgroup of $G$ and hence $\langle g \rangle$ and $\langle h \rangle$ permute.
Lemma

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Lemma

Let $G$ be an $\mathcal{X}$-group of infinite rank in which every subgroup is permutable or of finite rank. If $G$ has elements of infinite order, then every subgroup of $T(G)$ is normal in $G$. 
$x \in T(G)$, $y$ of infinite order. Then $\langle x \rangle \langle y \rangle \leq G$ and $\langle x \rangle \langle y \rangle \cap T(G) \trianglelefteq \langle x \rangle \langle y \rangle$. Then $y \in N_G(\langle x \rangle)$ so $\langle x \rangle \lhd G$. 
Proof of main theorem

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- $T(G)$ is abelian and $G/T(G)$ is a torsion-free abelian group of rank one.
Proof of main theorem

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- $T(G)$ is abelian and $G/T(G)$ is a torsion-free abelian group of rank one.

Write $A = B \times C$; $B, C$ have infinite rank. Every subgroup of $T(G)$ is normal so $B, C$ are normal subgroups of $G$. All subgroups of $G/B$ and $G/C$ are permutable so $T(G/B) = T(G)/B$ and $T(G/C) = T(G)/C$ are abelian. Then $T(G) \hookrightarrow T(G)/B \times T(G)/C$ implies that $T(G)$ is abelian.
If \( G/B \) and \( G/C \) are both abelian then \( G \hookrightarrow G/B \times G/C \) implies \( G \) is abelian. Thus one of \( G/B \) or \( G/C \) is nonabelian, say \( G/B \) is nonabelian. Then \( G/T(G) \cong (G/B)/T(G/B) \) is torsion-free abelian of rank one.
Proof of main theorem

If $G/B$ and $G/C$ are both abelian then $G \hookrightarrow G/B \times G/C$ implies $G$ is abelian. Thus one of $G/B$ or $G/C$ is nonabelian, say $G/B$ is nonabelian. Then $G/T(G) \cong (G/B)/T(G/B)$ is torsion-free abelian of rank one. $T(G)$ has infinite rank. We need to show that $\langle x \rangle \langle y \rangle T(G)$ is a group with all subgroups permutable. Since $\langle x, y \rangle T(G)/T(G)$ is finitely generated it is cyclic. Thus for the remainder of the proof we may assume that $G/T(G)$ is infinite cyclic.
Proof of main theorem

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\( T(G) \) has infinite rank. We need to show that \( \langle x \rangle \langle y \rangle T(G) \) is a group with all subgroups permutable. Since \( \langle x, y \rangle T(G)/T(G) \) is finitely generated it is cyclic. Thus for the remainder of the proof we may assume that \( G/T(G) \) is infinite cyclic.

In fact \( G \) is the semidirect product of \( T(G) \) by an infinite cyclic group \( \langle z \rangle \), and for every prime \( p \) there exists a \( p \)-adic unit \( r(p) \) with \( r(p) \equiv 1 \pmod{p} \) and \( r(2) \equiv 1 \pmod{4} \) such that \( a^z = a^{r(p)} \) for all \( a \in T(G)_p \), the \( p \)-component of \( T(G) \). Then the main theorem follows by the structure theorem for groups with all subgroups permutable.