Finite groups with a splitting automorphism

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Splitting automorphism

Let G be a group. An automorphism α of G is called a **splitting automorphism** if for every $x \in G$

$$xx^{\alpha}x^{\alpha^2}\dots x^{\alpha^{n-1}} = 1$$

where $|\alpha| = n$.

A fixed-point-free automorphism of a finite group is a splitting automorphism.

Splitting automorphism

In particular, if G is finite and $\alpha \in AutG$ fixed-point-free then the map

$$\begin{aligned} \psi : & G \longrightarrow G \\ & x \longrightarrow x^{-1}x^{\alpha} \end{aligned}$$

is surjective. Therefore, for every $g \in G$, there exists $x \in G$ with $g = x^{-1}x^{\alpha}$, so,

$$\begin{aligned} & gg^{\alpha}g^{\alpha^2}\dots g^{\alpha^{n-1}} = \\ & (x^{-1}x^{\alpha})(x^{-\alpha}x^{\alpha^2})\dots(x^{-\alpha^{n-1}}x^{\alpha^n}) = 1 \end{aligned}$$

Splitting automorphism

Let G be a non-perfect group. An element $a \in G$ is called an **anticentral element** if $aG' = a^G$.

Anticentral elements of finite order of a group G induce splitting automorphisms on G'.
Indeed, if a is anticentral in G with $|a| = n$, then for every $x \in G'$ the element xa^{-1} is conjugate to a^{-1}, which has order n. Then

$$xx^a x^{a^2} \ldots x^{a^{n-1}} = (xa^{-1})^n = 1.$$

Splitting automorphism

Example Let $G = UT(n,q)$ be the group of $n \times n$ upper triangular matrices with diagonal entries are equal to 1, where $n \geq 4$ and $q = p^k$ for some prime p.

Consider $a = \begin{pmatrix} 1 & 1 & & & & & \\ & 1 & 1 & & & & \\ & & 1 & . & & & \\ & & & . & . & & \\ & & & & . & . & \\ & & & & & 1 & 1 \\ & & & & & & 1 \end{pmatrix} \in G$. Namely, a is a regular unipotent in $GL(n,q)$ which is contained in $UT(n,q)$ that has all 1's in the first upper diagonal.

Splitting automorphism

Now, G' is the subgroup of G consisting of elements $x \in G$ such that $x_{12} = x_{23} = x_{34} = \ldots x_{(n-1)(n)} = 0$. Then, observe that $|G'| = |aG'| = q^{\frac{(n-2)(n-1)}{2}}$. But $|C_G(a)| = q^{n-1}$ where $|G| = q^{\frac{(n-1)(n)}{2}}$. So, $|a^G| = |aG'|$. Since $a^G \leq aG'$, they are equal, a is an anticentral element of G, so a is a splitting automorphism of G'.

Splitting automorphism

Remark Let G be a finite group.

- If α is a fixed-point-free automorphism of G then α is anticentral in $H = G\langle\alpha\rangle$.
- If a is an anticentral element of G, then a is a splitting automorphism of G'.

Splitting automorphism

Thompson proved that a finite group with a fixed-point-free automorphism of prime order is nilpotent. Kegel proved that the same is true for a finite group with a splitting automorphism of prime order.

Splitting automorphism

Rowley proved that a finite group with a fixed-point-free automorphism is solvable.

Ladisch proved that a finite group with an anticentral element is solvable

Splitting automorphism

It is natural to ask the following question:

Question 1 *Is a finite group with a splitting automorphism necessarily solvable?*

Splitting automorphism

By Kegel's result the answer is positive for splitting automorphisms of prime order. Moreover, Jabara proved that a finite group with a splitting automorphism of order 4 is solvable.

Splitting automorphism

However, one can see that the answer is negative in the full generality:
Example (Rowley) Observe that

$$\begin{aligned} \alpha :\ \mathbb{Z}_{31} &\longrightarrow \mathbb{Z}_{31} \\ x &\longrightarrow 11x \end{aligned}$$

is a fixed-point-free automorphism of the cyclic group $\mathbb{Z}_{31}$ of order 30.
Define $G = \mathbb{Z}_{31} \times A_5$ and consider

$$\begin{aligned} \phi :\ \mathbb{Z}_{31} \times A_5 &\longrightarrow \mathbb{Z}_{31} \times A_5 \\ (x, y) &\longrightarrow (x^{\alpha}, y). \end{aligned}$$

Splitting automorphism

One can observe that ϕ is a splitting automorphism of G of order 30, but G is not solvable.

Splitting automorphism

These kind of examples motivate the following question:
Question 2 Let n be natural number which is not divisible by the exponent of any non-abelian finite simple group.

Is a finite group with a splitting automorphism of order n necessarily solvable?

Splitting automorphism

In this talk, we will answer Question 2 partially, by proving the following result:

Theorem 1 (E.) *A finite group with a splitting automorphism of odd order is solvable.*

Splitting automorphism

In the proof, two basic properties of splitting automorphisms are used.

Proposition Let G be a group with an automorphism α of order dividing n, satisfying

$$xx^{\alpha}x^{\alpha^2}\dots x^{n-1}=1$$

for every $x \in G$. Then

1. $C_G(\alpha)$ has exponent dividing n.
2. For every $g \in G$, the element $g\alpha^{-1} \in G\langle\alpha\rangle$ has order dividing n.

Splitting automorphism

Proof of Theorem 1 Let G be a finite group with a splitting automorphism α of odd order n. Let R be the solvable radical of G. Clearly R is α-invariant. Moreover, the map given by $\overline{\alpha}(xR) = x^{\alpha}R$ for every $xR \in G/R$ is an automorphism of G/R of odd order, dividing n.

Splitting automorphism

For every $xR \in G/R$, one has

$$(xR)(xR)^{\overline{\alpha}}(xR)^{\overline{\alpha}^2}\dots(xR)^{\overline{\alpha}^{n-1}} = R.$$

Splitting automorphism

Therefore, we may assume $R = 1$, and G has an automorphism α of order dividing n, where n is odd, and

$$xx^{\alpha}x^{\alpha^2}\ldots x^{\alpha^{n-1}} = 1.$$

Splitting automorphism

Take a minimal normal subgroup M of G.

$$M \cong S \times S \times \ldots S$$

for some non-abelian simple group S. Take an orbit $S, S^{\alpha}, S^{\alpha^2} \ldots S^{\alpha^{t-1}}$. Clearly the length of the orbit divides the order of α.

Splitting automorphism

Here, if

$$(x_1, x_2^{\alpha}, x_3^{\alpha^2}, \ldots, x_t^{\alpha^{t-1}}) \in S \times S^{\alpha} \times S^{\alpha^2} \times \ldots \times S^{\alpha^{t-1}}$$

then

$$(x_1, x_2^{\alpha}, x_3^{\alpha^2}, \ldots, x_t^{\alpha^{t-1}})^{\alpha} = (x_t^{\alpha^t}, x_1^{\alpha}, x_2^{\alpha^2}, \ldots, x_{t-1}^{\alpha^{t-1}}).$$

Splitting automorphism

Therefore, $C_{S \times S^{\alpha} \times S^{\alpha^2} \times \ldots \times S^{\alpha^{t-1}}}(\alpha) \cong C_S(\alpha^t)$.

In this case, $\beta = \alpha^t$ is an automorphism of the non-abelian simple group S, which satisfies $ss^{\beta}s^{\beta^2} \ldots s^{\beta^{k-1}} = 1$ for all $s \in S$ where $|\beta|$ divides k, and k divides $\frac{n}{t}$.

Splitting automorphism

Now we need to show that a finite non-abelian simple group S can not have an automorphism of dividing k where $ss^{\beta}s^{\beta^2}\dots s^{\beta^{k-1}} = 1$ for each $s \in S$ and k is odd.

Splitting automorphism

Let S be a finite non-abelian simple group with an automorphism β with $ss^{\beta}s^{\beta^2}\ldots s^{\beta^{k-1}} = 1$ for each $s \in S$. Then, for each $s \in S$, the element $s\beta^{-1} \in AutS$ has odd order. If β is an inner automorphism, multiplying each element of S with β^{-1} is a bijection of S. By Feit-Thompson Theorem, S has even exponent, so β can not be an inner automorphism.

Splitting automorphism

Then, S has an outer automorphism of odd order, which is only possible when S is a simple group of Lie type.

Splitting automorphism

Then, there exists a simple linear algebraic group $\overline{S}$ over an algebraically closed field of characteristic p and a Frobenius map σ over $\overline{S}$ such that

$$S = O^{p'}(\overline{S}_\sigma)$$

where $\overline{S}_\sigma$ denotes the group of fixed-points of σ in $\overline{S}$.

Splitting automorphism

By a result of Steinberg, any automorphism of a finite simple group of Lie type is a product of an inner-diagonal automorphism, a field automorphism and a graph automorphism.

Namely, $\beta = g\phi\delta$ where $g \in \overline{S}_\sigma$. Also, ϕ is a Frobenius map on $\overline{S}$ with $\phi^m = \sigma$ for some m, and δ is induced by a symmetry of the Dynkin diagram.

Splitting automorphism

We need to analyse the cases seperately:
If β is an inner-diagonal automorphism, say
$\beta = g \in \overline{S}_\sigma \backslash O^{p'}(\overline{S}_\sigma)$,

then there exists an element $x \in S$ such that xg^{-1} is diagonal.

Now, take an involution $i \in C_S(xg^{-1})$, so, $ix \in S$ and hence ixg^{-1} must have order dividing $|\beta|$ which is odd. But, since i and xg^{-1} commute, it has even order.

Splitting automorphism

Now, assume $\beta = g\phi$ where ϕ is a field automorphism. Here, $C_S(\beta) = O^{p'}(C_{\overline{S}}(\sigma, g\phi)) \geq C_{O^{p'}(\overline{S}_\phi)}(g)$, which contains an involution (or an element of order not dividing $|g|$) unless g is a regular unipotent or regular semisimple element of $\overline{S}_\phi$.

In both cases, one can find an element $x \in S$ such that $x\beta^{-1}$ has even order.

Splitting automorphism

Finally one has to analyse the case $\beta = g\phi\delta$, where δ is a non-trivial graph automorphism. This happens for the simple groups of type D_4, since β has odd order.

Then, one can pick an involution from the root subgroup fixed by δ, to construct an element $x \in S$, such that $x\beta^{-1}$ has even order.

Splitting automorphism

Therefore, a finite simple group can not have an automorphism of odd order n satisfying

$$xx^{\alpha}\dots x^{\alpha^{n-1}} = 1$$

and hence, a finite group with a splitting automorphism of odd order is solvable.

This result has a consequence, related to some earlier work of the speaker.

Anticentral elements

Recall that an element $a \in G$ is called anticentral if $aG' = a^G$. Ladisch proved that a finite group with an anticentral element is solvable.

Question 2 *Is every locally finite group with an anticentral element locally solvable?*

Anticentral elements

Theorem 2 (E.) *Let G be a group with an anticentral element a of order m such that G' is a periodic $\mathbb{F}$-linear group where $\mathbb{F}$ has characteristic p. Then one of the following cases occurs:*

1. *$C_{G'}(a)$ is finite and G is solvable.*
2. *$C_{G'}(a)$ has an infinite abelian subgroup of exponent p^k where p^k divides m.*

Anticentral elements

By using Theorem 1, the following result about locally finite groups with an anticentral element, follows easily: **Corollary** A locally finite group with an anticentral element of odd order is locally solvable.