Covering monolithic groups with proper subgroups

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These considerations led Cohn in 1994 to define for every group G:

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Theorem (Neumann 1954)

If G is a group with finite covering number then there exists $N \trianglelefteq G$ such that G/N is finite and $\sigma(G) = \sigma(G/N)$.

From now on every considered group is assumed to be finite.

Example

If p is a prime number then $\sigma(C_p \times C_p) = p + 1$. Indeed, $C_p \times C_p$ has exactly $p + 1$ maximal subgroups, all of them isomorphic to C_p and pairwise intersecting in the identity subgroup, so they cover $1 + (p - 1)(p + 1) = p^2$ elements.

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Actually the solvable case has been completely worked out:

Theorem (Tomkinson, [3])

Let G be a finite solvable group. Then

$$\sigma(G) = |H/K| + 1$$

where H/K is the smallest chief factor of G with more than one complement in G/K.

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This suggests to consider the quotients G/N such that $\sigma(G) = \sigma(G/N)$, and leads to the following:

Definition (σ-elementary groups)

A group G is said to be "σ-elementary" if

$$\sigma(G) < \sigma(G/N)$$

for every $1 \neq N \trianglelefteq G$. We say that G is "n-elementary" if G is σ-elementary and $\sigma(G) = n$.

Clearly if G is any finite group then there exists $N \trianglelefteq G$ such that G/N is σ-elementary and $\sigma(G) = \sigma(G/N)$.

Example

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Recall that the "**socle**" of a group is the subgroup generated by its minimal normal subgroups, and a group is called "**monolithic**" if it has only one minimal normal subgroup.

Example

Tomkinson's result implies that if G is a non-abelian solvable σ-elementary group then G is monolithic, $G/\operatorname{soc}(G)$ is cyclic and

$$\sigma(G) = |\operatorname{soc}(G)| + 1.$$

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Theorem (Lucchini, Detomi [2] Corollary 14)

Let G be a non-abelian σ-elementary group and let $N_1, \ldots, N_k$ be the minimal normal subgroups of G.

Then there exist epimorphic images $X_1, \ldots, X_k$ of G with the property that X_i is a primitive monolithic group with socle isomorphic to N_i for $i = 1, \ldots, k$ and G is a subdirect product of $X_1, \ldots, X_k$: the natural homomorphism

$$G \to X_1 \times \ldots \times X_k$$

is injective.

This and other results led Lucchini and Detomi to formulate the following conjecture:

Conjecture (Lucchini, Detomi [2])

Every non-abelian σ-elementary group is monolithic.

If the non-abelian group G is a direct product of two non-trivial subgroups then it is not σ-elementary. This is a consequence of the following result:

Theorem (Lucchini A., G 2010 [4])

Let $\mathcal{M}$ be a minimal cover of a direct product $G = H_1 \times H_2$ of two finite groups. Then one of the following holds:

1. $\mathcal{M} = \{X \times H_2 \mid X \in \mathcal{X}\}$ *where $\mathcal{X}$ is a minimal cover of H_1. In this case* $\sigma(G) = \sigma(H_1)$.
2. $\mathcal{M} = \{H_1 \times X \mid X \in \mathcal{X}\}$ *where $\mathcal{X}$ is a minimal cover of H_2. In this case* $\sigma(G) = \sigma(H_2)$.
3. *There exist* $N_1 \trianglelefteq H_1$, $N_2 \trianglelefteq H_2$ *with* $H_1/N_1 \cong H_2/N_2 \cong C_p$ *and $\mathcal{M}$ consists of the maximal subgroups of $H_1 \times H_2$ containing $N_1 \times N_2$. In this case* $\sigma(G) = p + 1$.

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σ	groups
3	$C_2 \times C_2$
4	$C_3 \times C_3, \mathrm{Sym}(3)$
5	$\mathrm{Alt}(4)$
6	$C_5 \times C_5, D_{10}, AGL(1,5)$
7	$\emptyset$
8	$C_7 \times C_7, D_{14}, 7:3, AGL(1,7)$
9	$AGL(1,8)$
10	$3^2:4, AGL(1,9), \mathrm{Alt}(5)$
11	$\emptyset$
12	$C_{11} \times C_{11}, 11:5, D_{22}, AGL(1,11)$
13	$\mathrm{Sym}(6)$
14	$C_{13} \times C_{13}, D_{26}, 13:3, 13:4, 13:6, AGL(1,13)$
15	$SL(3,2)$
16	$\mathrm{Sym}(5), \mathrm{Alt}(6)$
17	$2^4:5, AGL(1,16)$
18	$C_{17} \times C_{17}, D_{34}, 17:4, 17:8, AGL(1,17)$
19	$\emptyset$
20	$C_{19} \times C_{19}, AGL(1,19), D_{38}, 19:3, 19:6, 19:9$
21	$\emptyset$
22	$\emptyset$
23	M_{11}
24	$C_{23} \times C_{23}, D_{46}, 23:11, AGL(1,23)$
25	$\emptyset$

By using similar techniques it is possible to prove that:

Theorem

Let G be a non-abelian σ-elementary group, and assume that $\sigma(G) \leq 56$. Then G is either affine or almost simple.

Moreover we have $\sigma(A_5 \wr C_2) = 4 \cdot 5 + 6 \cdot 6 + 1 = 57$.

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Theorem

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Moreover we have $\sigma(A_5 \wr C_2) = 4 \cdot 5 + 6 \cdot 6 + 1 = 57$.

Note that $G := A_5 \wr C_2$ is the smallest monolithic group which is neither affine nor almost simple. A minimal covering of this group consists of the socle plus subgroups of the form

$$N_G(M \times M^a)$$

where M varies in the covering of A_5 consisting of the six normalizers of the Sylow 5-subgroups and four point stabilizers, and a varies in A_5. This argument has been generalized and the following results were obtained.

Let G be a monolithic group with non-abelian socle

$$S_1 \times \ldots \times S_m = S^m,$$

where $S = \mathrm{Alt}(n)$, $n \geq 5$ and $G/\mathrm{soc}(G)$ is cyclic. Let

$$X := N_G(S_1)/C_G(S_1).$$

X is an almost-simple group with socle isomorphic to S, and G embeds in the wreath product $X \wr \mathrm{Sym}(m)$.

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The coverings we prove to be minimal in the following theorems consist of the subgroups containing the socle together with the subgroups of the form

$$N_G(M \times M^{a_2} \times \ldots \times M^{a_m})$$

where $a_1, \ldots, a_m$ vary in X and M varies in a suitable family of subgroups of X which covers xS, where $\langle xS \rangle = X/S$. By $\omega(x)$ we denote the number of prime divisors of the integer x.

Theorem (Maróti, G 2010 [6])

Suppose $X = S = \mathrm{Alt}(n)$. Then the following holds.

1. *If* $12 < n \equiv 2 \mod (4)$ *then*

$$\sigma(G) = \omega(m) + \sum_{i=1,\ i\ odd}^{(n/2)-2} \binom{n}{i}^m + \frac{1}{2^m}\binom{n}{n/2}^m.$$

2. *If* $12 < n \not\equiv 2 \mod (4)$ *then*

$$\omega(m) + \frac{1}{2}\sum_{i=1,\ i\ odd}^{n} \binom{n}{i}^m \leq \sigma(G).$$

3. *Suppose n has a prime divisor at most $\sqrt[3]{n}$. Then*

$$\sigma(G) \sim \omega(m) + \min_{\mathcal{M}} \sum_{M \in \mathcal{M}} |S : M|^{m-1} \ as\ n \to \infty.$$

Theorem (G 2011 [7])

Suppose $X = \mathrm{Sym}(n)$. *Then the following holds.*

1. *Suppose that* $n \geq 7$ *is odd and* $(n, m) \neq (9, 1)$. *Then*

$$\sigma(G) = \omega(2m) + \sum_{i=1}^{(n-1)/2} \binom{n}{i}^m.$$

2. *Suppose that* $n \geq 8$ *is even. Then*

$$\left(\frac{1}{2}\binom{n}{n/2}\right)^m \leq \sigma(G) \leq \omega(2m) + \left(\frac{1}{2}\binom{n}{n/2}\right)^m + \sum_{i=1}^{[n/3]} \binom{n}{i}^m.$$

In particular $\sigma(G) \sim \left(\frac{1}{2}\binom{n}{n/2}\right)^m$ *as* $n \to \infty$.

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Proposition (Lucchini, Detomi [2], Proposition 16)

Let X be a monolithic primitive group with socle N. If Ω is an arbitrary union of cosets of N in X define $\sigma_\Omega(X)$ to be the smallest number of supplements of N in X needed to cover Ω. Define

$$\sigma^*(X) := \min\{\sigma_\Omega(X) \mid \Omega = \bigcup_i \omega_i N,\ \langle \Omega \rangle = X\}.$$

Let now G be a non-abelian σ-elementary group,

$$G \leq_{subd} X_1 \times \ldots \times X_k.$$

Then

$$\sigma^*(X_1) + \ldots + \sigma^*(X_k) \leq \sigma(G).$$

In particular, since $X_1, \dots, X_k$ are epimorphic images of G, for every $i \in \{1, \dots, k\}$ we have

$$\sigma^*(X_1) + \ldots + \sigma^*(X_k) \leq \sigma(G) \leq \sigma(X_i).$$

This means that the study of the quantity

$$\sigma(X) - \sigma^*(X)$$

for X a primitive monolithic group gives informations about the validity of the conjecture.

The ideas employed in Theorems 7 and 8 can be adapted to prove that if $X/\operatorname{soc}(X)$ is cyclic and $\operatorname{soc}(X) = \mathrm{Alt}(n)^m$ then in most cases the quantity $\sigma(X) - \sigma^*(X)$ is "small". But this is still a work in progress.

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