Polynomial identities and codimension growth

Antonio Giambruno

Dipartimento di Matematica ed Informatica

Università di Palermo

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A= non necessarily associative algebra over F.

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Recall: $f(x_1, \ldots, x_n) \in F\{X\}$ is a polynomial identity of A if $f(a_1, \ldots, a_n) = 0$ for all $a_i \in A$.

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For every $n \geq 1$, let P_n be the space of multilinear polynomials in $x_1, \ldots, x_n$.

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- The number of distinct arrangements of parentheses on a monomial of length n is the Catalan number $\frac{1}{n}\binom{2n-2}{n-1}$.

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$c_n(F\{X\}) = \dim_F P_n = \binom{2n-2}{n-1}(n-1)!$.

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- If $F\langle X\rangle$ is the free associative algebra, $c_n(F\langle X\rangle) = n!$

- For $L\langle X\rangle$ = the free Lie algebra, $c_n(L\langle X\rangle) = (n-1)!$.

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• For a Lie or Jordan or alternative PI-algebra, there is no intermediate growth of the codimensions.

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- $A = F[x]$

  $Id(A) = \langle [x_1, x_2] \rangle_T$

  $$c_n(A) = 1, \text{ for all } n \geq 1, \; exp(F[x]) = 1.$$

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  $$c_n(M_2(F)) \underset{n \to \infty}{\simeq} \frac{4^{n-1}}{n\sqrt{\pi n}},$$

  $exp(M_2(F)) = 4$.

- $A = \begin{pmatrix} G & G \\ 0 & G_0 \end{pmatrix}$ where $G = G_0 \oplus G_1$,

$G_0 = \text{span}_F\{e_{i_1} \cdots e_{i_{2t}} \mid 1 \leq i_1 < \cdots < i_{2t}\}$,
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OPEN PROBLEM. In case the sequence of codimensions is exponentially bounded, does $exp(A) = \lim_{n\to\infty} \sqrt[n]{c_n(A)}$ exist? Is it an integer?

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Theorem Let A be a finite dimensional associative algebra over an algebraically closed field F. Then $exp(A) = \dim A$ if and only if A is simple.

Let A be a finite dimensional algebra over F.
let $\alpha(x, y)$ be a fixed linear combination of elements of the type $T_u T'_v, T_{uv}$, where T, T' are left or right multiplication and $\{u, v\} = \{x, y\}$.
Denote by $\langle x, y\rangle = \text{tr}(\alpha(x, y))$ the bilinear form determined by α.

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G.-Zaicev (2012). Let A be a finite dimensional simple algebra over a field of characteristic zero. Then exp$(A) = \lim_{n\to\infty} \sqrt[n]{c_n(A)}$ exists and exp$(A) \leq \dim A$.

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$b(t)$ is the Lie superalgebra of $2t \times 2t$ matrices over F of the type

$$\begin{pmatrix} A & B \\ C & -A^T \end{pmatrix},$$

where $A, B, C \in M_t(F)$, $B^T = B, C^T = -C$ and $trA = 0$.

Fix $t \geq 3$ and write $b(t) = L = L_0 \oplus L_1$ where

$$L_0 = \{ \begin{pmatrix} A & 0 \\ 0 & -A^T \end{pmatrix} \mid A \in M_t(F), tr(A) = 0\},$$

and

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The algebra L is a Lie superalgebra with $\mathbb{Z}_2$-grading $L = L_0 \oplus L_1$ if we define a product $[\ ,\ ]$ of two homogeneous elements $x, y \in L$ as

$$[x, y] = xy - (-1)^{|x||y|} yx.$$

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If $Id(A)$ is a T-ideal, $P_n \cap Id(A)$ is invariant under this action. Hence

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Let $\chi_n(A)$ be its S_n-character (called the n-th cocharacter of A).

By complete reducibility write

$$\chi_n(A) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda$$

where for a partition λ of n, χ_λ is the irreducible S_n-character associated to λ and $m_\lambda \geq 0$ is the corresponding multiplicity.

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G.-Mishchenko-Zaicev If $\dim A = d < \infty$, then the multiplicities m_λ are polynomially bounded ($\sum_{\lambda \vdash n} m_\lambda \leq d(n+1)^{d^2+d}$).

Amitsur-Regev. If A is a PI-algebra, there exist integers k, l such that

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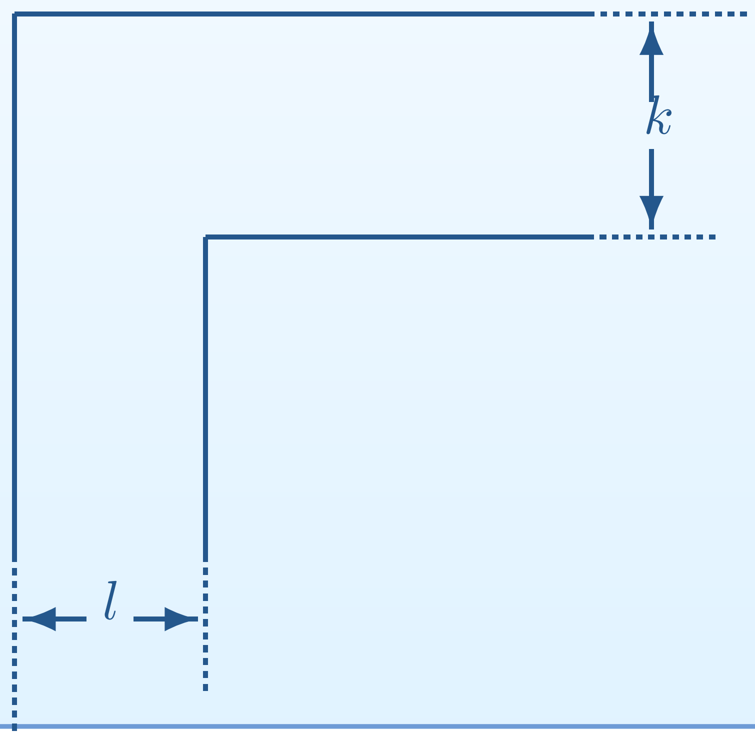
Thus $H(k,l)$ = the set of all diagrams whose shape lies in the hook shaped region



$$c_n(A) = \sum_{\lambda \vdash n} m_\lambda d_\lambda, \tag{1}$$

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Theorem . If $\dim A = d < \infty$, then

$$c_n(A) = \sum_{\substack{\lambda \vdash n \\ \lambda \in H(d,0)}} m_\lambda d_\lambda$$

it follows that asymptotically

$$\sqrt[n]{c_n(A)} \simeq \sqrt[n]{(d_\lambda)_{max}}, \tag{4}$$

where

$$(d_\lambda)_{max} = \text{max}\{d_\lambda \mid \lambda \vdash n \text{ with } m_\lambda \neq 0 \text{ in } (1)\}. \tag{5}$$

Given a partititon $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash n$ of height $k \leq d$, we write $\lambda = (\lambda_1, \ldots, \lambda_d)$ with $\lambda_{k+1} = \ldots = \lambda_d = 0$.

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Then we define

$$\Phi(\lambda) = \frac{1}{z_1^{z_1} \ldots z_d^{z_d}},$$

where $z_1 = \frac{\lambda_1}{n}, \ldots, z_d = \frac{\lambda_d}{n}$ and we set $\lambda_j^{\lambda_j} = 1$ if $\lambda_j = 0$. Hence $0 \leq z_1, \ldots, z_d \leq 1$ and $z_1 + \cdots + z_d = 1$.

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Remark. For n large and $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash n$, with $k \leq d$, we have

$$\frac{\Phi(\lambda)^n}{n^{d^2+d}} \leq d_\lambda \leq n\Phi(\lambda)^n.$$

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Thus $\sqrt[n]{c_n(A)} \simeq \Phi(\lambda)$ where $\lambda \vdash n$ is such that $d_\lambda = (d_\lambda)_{max}$.

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Then we define

$$\Phi(\lambda) = \frac{1}{z_1^{z_1} \cdots z_d^{z_d}},$$

where $z_1 = \frac{\lambda_1}{n}, \ldots, z_d = \frac{\lambda_d}{n}$ and we set $\lambda_j^{\lambda_j} = 1$ if $\lambda_j = 0$. Hence $0 \leq z_1, \ldots, z_d \leq 1$ and $z_1 + \cdots + z_d = 1$.

Remark. For n large and $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash n$, with $k \leq d$, we have

$$\frac{\Phi(\lambda)^n}{n^{d^2+d}} \leq d_\lambda \leq n\Phi(\lambda)^n.$$

Thus $\sqrt[n]{c_n(A)} \simeq \Phi(\lambda)$ where $\lambda \vdash n$ is such that $d_\lambda = (d_\lambda)_{max}$.