Finite simple quotients of groups satisfying property (T).

Mikhail Ershov, Andrei Jaikin-Zapirain and Martin Kassabov

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Expanders

Definition: ϵ-Expander

For $0 < \epsilon \in \mathbb{R}$ a graph

$$X = \left(\underset{\text{vertices}}{\underset{\uparrow}{V}} , \underset{\text{edges}}{\underset{\uparrow}{E}} \right)$$

is ϵ-expander if $\forall A \subset V$, with $|A| \leq \frac{|V|}{2}$, $|\partial A| \geq \epsilon |A|$,
where $\partial A =$ boundary of $A = \{v \in V : \ dist(v, A) = 1\}$

Definition: Expander family

A family $\{X_i\}_{i\in\mathbb{N}}$ of k-regular finite graphs is a *family of expanders* if $\exists \epsilon > 0$ such that $\forall i \ X_i$ is an ϵ-expander and $|X_i| \to \infty$.

Expanders are simultaneously sparse and highly connected.

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Network constructions
Error-correcting codes
Cryptography
Number Theory
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Expander families exist:
1973: Pinsker: Using counting arguments

For applications one wants explicit constructions.

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Property (T)

Definition: Property (T) (1967, Kazhdan)

Let $\Gamma = \langle S \rangle$ with S finite. Then Γ has *property* (T) if $\exists \epsilon > 0$ such that

$\forall \rho : \Gamma \to U(\mathcal{H})$ (where $\mathcal{H}$ is a Hilbert space) and $\forall v \in \mathcal{H}_0^{\perp}$ (where $\mathcal{H}_0$ is the space of Γ-invariant vectors of $\mathcal{H}$)

$$\|\rho(s)v - v\| \geq \epsilon \|v\|, \forall s \in S.$$

There are not almost Γ-invariant vectors in $\mathcal{H}_0^{\perp}$.

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Construction of expanders using property (T)

Let $\Gamma = \langle S \rangle$ (with $S = S^{-1}$ finite) is infinite and residually finite and satisfies property (T)

Let $\{\Gamma_i\}_{i \in \mathbb{N}}$ a family of normal subgroups of Γ of finite index such that $|\Gamma/\Gamma_i|$ tends to infinity.

Then the Cayley graphs $X_i = \mathrm{Cay}(\Gamma/\Gamma_i; S)$ form a family of expanders.

Definition: Property tau (1989 Lubotzky, Zimmer)

Γ has *property tau* if $\exists \epsilon > 0$ such that

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Family of group expanders

Definition: family of group expanders

A family $\{G_i\}$ of finite groups is a *family of expanders* if $\exists k \in \mathbb{N}$ and $\epsilon > 0$ such that every group G_i has a symmetric subset S_i of k generators for which $\mathrm{Cay}(G_i; S_i)$ is an ϵ-expander.

Example: If Γ is a finitely generated group satisfying property (T) or tau, then

$$\{\text{finite quotients of } \Gamma\}$$

is a family of expanders.

Conjecture (1989 Babai, Kantor and Lubotzky)

The family of all the finite (nonabelian) simple groups is a family of group expanders.

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Theorem (Kassabov, Nikolov, Lubotzky, Breulliard, Green, Tao)

There exists $k \in \mathbb{N}$ and $\epsilon > 0$ such that every every non-abelian finite simple group G has a set S of k generators for which $\mathrm{Cay}(G; S)$ is an ϵ-expander.

2005 M. Kassabov: $PSL_n(\mathbb{F}_q)(n \geq 3)$ and Alt_n
2005 N. Nikolov: classical groups of large rank
2005 A. Lubotzky: $PSL_2(\mathbb{F}_q)$ and simple groups of Lie type of bounded rank with the exception of the Suzuki groups
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Mother group

Definition: mother group

Let $\mathcal{F}$ be a family of groups. We say that a group Γ is a mother group for $\mathcal{F}$ if Γ maps onto every group from $\mathcal{F}$.

Question.

Let $\mathcal{F}$ be a family of finite simple groups. Does $\mathcal{F}$ have a finitely generated mother group satisfying Kazhdan's property (T) or property tau?

Example: $SL_3(\mathbb{Z})$ is a mother group of $\{PSL_3(\mathbb{F}_p)\}$.

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Mother group

Conjecture

1. The family of all the non-abelian finite simple groups has a finitely generated mother group having property tau.
2. A family $\mathcal{F}$ of non-abelian finite simple groups has a mother group having property (T) if and only if only finitely many finite simple groups of Lie type of rank 1 belongs to $\mathcal{F}$.

Why do we exclude the groups of Lie type of rank 1 in (2)?

Theorem (folklore)

Let Γ maps on infinitely many $PSL_2(\mathbb{F}_q)$. Then Γ does not have property (T).

$SL_2(\mathbb{Z}[1/2])$ has property tau and maps onto $\{PSL_2(\mathbb{F}_p)\}$

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Main result

Theorem

There exists a group Γ satisfying property (T) such that every finite simple group of Lie type of rank at least 2 is a quotient of Γ.

Main tool:

A method that allows to prove that some groups graded by root systems have property (T).

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Root systems

Definition

Let E be real vector space. A finite non-empty subset Φ of E is called a *root system in E* if

(a) Φ spans E;

(b) Φ does not contain 0;

(c) Φ is closed under inversion, that is, if $\alpha \in \Phi$ then $-\alpha \in \Phi$.

The dimension of E is called the *rank of* Φ.

Definition

A root system Φ in a space E will be called *classical* if E can be given the structure of a Euclidean space such that

(a) For any $\alpha, \beta \in \Phi$ we have $\frac{2(\alpha,\beta)}{(\beta,\beta)} \in \mathbb{Z}$;

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Root systems

Remark: Every irreducible classical root system is isomorphic to one of the following: $A_n, B_n (n \geq 2), C_n (n \geq 3), BC_n (n \geq 1), D_n (n \geq 4), E_6, E_7, E_8, F_4, G_2$.

A_2

B_2

G_2

BC_2

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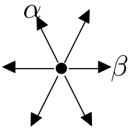
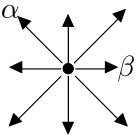
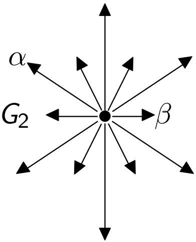
A_2



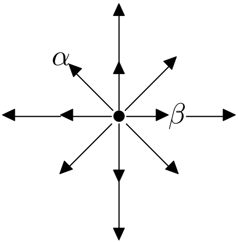
B_2



G_2



BC_2



Groups graded by root systems

Definition: Φ-grading

Φ a root system, G a group. A *Φ-grading of G* is a collection of subgroups $\{X_\alpha\}_{\alpha\in\Phi}$ of G, called *root subgroups* such that

(i) $\{X_\alpha\}_{\alpha\in\Phi}$ generate G

(ii) For any $\alpha, \beta \in \Phi$, with $\alpha \notin \mathbb{R}_{<0}\beta$, we have

$$[X_\alpha, X_\beta] \subseteq \langle X_\gamma \mid \gamma = a\alpha + b\beta \in \Phi,\ a, b \geq 1\rangle$$

Informal definition: graded cover

A graded cover of a grading $\{X_\alpha\}_{\alpha\in\Phi}$ is the quotient of the free product of $\{X_\alpha\}_{\alpha\in\Phi}$ by the normal subgroup generated by relations that appear in (ii) of the previous definition

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Theorem (2010, Ershov, Jaikin, Kassabov)

Let Φ be a *regular* (for example, classical) root system of rank at least two. Let G be a finitely generated group and $\{X_\alpha\}_{\alpha\in\Phi}$ its Φ-grading. Assume that

1. the grading $\{X_\alpha\}$ is *strong* and
2. the pair $(G, \cup_{\alpha\in\Phi} X_\alpha)$ has relative Kazhdan property,

then G has property (T).

2008 Ershov, Jaikin: the case of groups with A_2-grading

Theorem(2008, Ershov, Jaikin)

Let R be a finitely generated ring and $n \geq 3$. Let $G = EL_n(R)$, that is, the subgroup of $GL_n(R)$ generated by elementary matrices. Then G has property (T).

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The simple groups of Lie type have natural gradings

Type of simple group	Classical group	Graded cover
$A_n(q)$	$PSL_{n+1}(\mathbb{F}_q)$	$St_{A_n}(\mathbb{F}_q)$
$B_n(q)$ (q is odd)	$\Omega_{2n+1}(\mathbb{F}_q)$	$St_{B_n}(\mathbb{F}_q)$
$C_n(q)$ ($q \neq 2$ if $n = 2$)	$PSp_{2n}(\mathbb{F}_q)$	$St_{C_n}(\mathbb{F}_q)$
$D_n(q)$	$P\Omega^{+}_{2n}(\mathbb{F}_q)$	$St_{D_n}(\mathbb{F}_q)$
$\Phi(q)$ ($\Phi = G_2, E_n$ or F_4 and $q \neq 2$ if $\Phi = G_2$)		$St_{\Phi}(\mathbb{F}_q)$

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$C_n(q)$ ($q \neq 2$ if $n = 2$)	$PSp_{2n}(\mathbb{F}_q)$	$St_{C_n}(\mathbb{F}_q)$
$D_n(q)$	$P\Omega^+_{2n}(\mathbb{F}_q)$	$St_{D_n}(\mathbb{F}_q)$
$\Phi(q)$ ($\Phi = G_2, E_n$ or F_4 and $q \neq 2$ if $\Phi = G_2$)		$St_{\Phi}(\mathbb{F}_q)$

Groups graded by root systems

Type of simple group	Classical group	Graded cover
${}^2A_{2m-1}(q)$	$PSU_{2m}(\mathbb{F}_q)$	$St^1_{C_m}(\mathbb{F}_{q^2}, *)$,
${}^2A_{2m}(q)$	$PSU_{2m+1}(\mathbb{F}_q)$	$St_{BC_m}(\mathbb{F}_{q^2}, *)$,
${}^2D_m(q)$	$P\Omega^-_{2m}(\mathbb{F}_q)$	$St^1_{B_{m-1}}(\mathbb{F}_{q^2}, Id, \sigma)$
${}^3D_4(q)$		$St_{G_2}(\mathbb{F}_{q^3}, \sigma)$,
${}^2E_6(q)$		$St_{F_4}(\mathbb{F}_{q^2}, \sigma)$,

${}^2F_4(2^k)(k \geq 1)$ is graded and its graded cover is $St_{{}^2F_4}(\mathbb{F}_{2^{2k+1}})$

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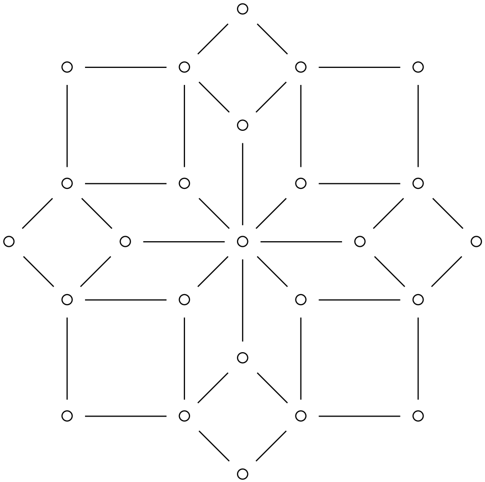

Main result

Theorem

There exists a group Γ satisfying property (T) such that every finite simple group of Lie type of rank at least 2 is a quotient of Γ.

Bounded rank case

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For each r, there exists a group Γ_r satisfying property (T) such that every finite simple group of Lie type of rank at least 2 and at most r is a quotient of Γ_r.

Proof in the non-twisted case:

We fix a root system Φ.

$St_\Phi(\mathbb{F}_q)$ is the graded cover of the simple group $\Phi(q)$ (with a finite number of exceptions).

$St_\Phi(\mathbb{Z}[t])$ maps onto $St_\Phi(\mathbb{F}_q)$.

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Mother group with property (T) for $\{PSL_n(\mathbb{F}_q)\}$

$A_2 = \{\pm\alpha, \pm\beta, \pm(\alpha+\beta)\}$

A_2-grading of $SL_n(\mathbb{F}_q)$, with $n = k + l + m$:

$$X_\alpha = \begin{pmatrix} 1 & \mathrm{Mat}_{k\times l}(\mathbb{F}_q) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, X_\beta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \mathrm{Mat}_{l\times m}(\mathbb{F}_q) \\ 0 & 0 & 1 \end{pmatrix}$$

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$$EL_3(\mathrm{Mat}_{k\times k}(\mathbb{F}_q)) \cong SL_{3k}(\mathbb{F}_q)$$

Let $R = \mathbb{Z} < x, y >$, then $EL_3(R)$ maps onto $EL_3(\mathrm{Mat}_{k\times k}(\mathbb{F}_q))$.

This proves that $\{PSL_{3k}(\mathbb{F}_q)\}$ has a mother group with property (T).

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Mother group with property (T) for symplectic groups

Let $J = \begin{pmatrix} 0 & 0 & 0 & I_k \\ 0 & 0 & I_k & 0 \\ 0 & -I_k & 0 & 0 \\ -I_k & 0 & 0 & 0 \end{pmatrix} \in \mathrm{Mat}_{4k}(\mathbb{F}_q)$

$$Sp_{4k}(\mathbb{F}_q) = \{A \in GL_{4k}(\mathbb{F}_q) : \; A^{transp} JA = J\}$$

There is no a symplectic group over non-commutative ring.

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C_2-grading of $Sp_{4k}(\mathbb{F}_q)$ $(A, B = B^{transp} \in \mathrm{Mat}_k(\mathbb{F}_q)\})$

$$X_{e_1-e_2} = \left\{ \begin{pmatrix} 1 & A & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -{}^tA \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}, \; X_{e_1+e_2} = \left\{ \begin{pmatrix} 1 & 0 & A & 0 \\ 0 & 1 & 0 & {}^tA \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\},$$

$$X_{2e_1} = \left\{ \begin{pmatrix} 1 & 0 & 0 & B \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}, \; X_{2e_2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & B & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \}$$

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Then G has property (T) (we prove it using our method) and G maps onto $Sp_{4k}(\mathbb{F}_q)$ ($\mathrm{Mat}_k(\mathbb{F}_q)$ can be generated as a ring by two symmetric matrices)

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Open questions

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Assume that G maps on infinitely many twisted Lie groups of rank 1:

$$^2A_2(q) = PSU_3(\mathbb{F}_q),\ \text{Suzuki groups }\ ^2B_2(2^{2n+1})$$
$$\text{or Ree groups }\ ^2G_2(3^{2n+1}).$$

Is it true that G can not have property (T)?

Question 2

Is there a finitely generated group with property tau that maps on all $PSL_2(\mathbb{F}_q)$?

Question 3

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