# Automorphisms of finite $p$-groups admitting a partition

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outside a proper subgroup all elements are of order $p$,

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(But there is no bound for the exponent of a $p$-group with a partition.)

# Splitting automorphism approach

Splitting automorphism approach of condition (c) turned out to be most efficient. Recall:

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All groups with a splitting automorphism of order $p$ form a variety of groups with operators defined by the laws $(*)$.

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As a corollary, a positive solution for the Hughes problem
was obtained for "almost all" finite $p$-groups.

# Nilpotency class depending on automorphisms

EKh–Shumyatsky, 1995: if a finite group $G$ of exponent $p$ admits a soluble group of automorphisms $A$ of coprime order such that the fixed-point subgroup $C_G(A)$ is soluble of derived length $d$, then $G$ is nilpotent of $(p, d, |A|)$-bounded class.

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### Theorem 1

Suppose that a finite $p$-group $P$ with a partition admits a soluble group of automorphisms $A$ of coprime order such that $C_P(A)$ has derived length $d$. Then any maximal subgroup of $P$ containing $H_p(P)$ is nilpotent of $(p, d, |A|)$-bounded class.

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Note: the nilpotency class of the whole group $P$ cannot be bounded.

The bound for the nilpotency class of that maximal subgroup can be chosen the same as in EKh–Shumyatsky-95 for groups of exponent $p$.

# Exponent

## Theorem 2

If a finite $p$-group $P$ with a partition admits a group of automorphisms $A$ that acts faithfully on $P/H_p(P)$, then the exponent of $P$ is bounded in terms of the exponent of $C_P(A)$.

# Frobenius groups of automorphisms

## Corollary

Suppose that a finite group $G$ admits a Frobenius group of automorphisms $FH$ with cyclic kernel $F = \langle \varphi \rangle$ of prime order $p$ such that $\varphi$ is a splitting automorphism, that is, $xx^{\varphi}x^{\varphi^2}\cdots x^{\varphi^{p-1}} = 1$ for all $x \in G$.

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(a) If $C_G(H)$ is soluble of derived length $d$, then $G$ is nilpotent of $(p,d)$-bounded class.

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Plus $p$-bounded nilpotency class of $G_{p'}$ $\Rightarrow$ exponent of $G_{p'}$ is bounded in terms of $p$ and exponent of $C_G(H)$.
So in (b) it remains to consider $G_p$. The result follows from Theorem 2 applied to $P = G_p\langle\varphi\rangle$ and $A = H$.

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Question remains open for the exponent, as well as for the derived length.

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**Theorem 1′**

Suppose that a soluble group $FA$ with normal Sylow $p$-subgroup $F = \langle \varphi \rangle$ of order $p$ and Hall $p'$-subgroup $A$ acts by automorphisms on a finite $p$-group $G$ in such a manner that $\varphi$ is a splitting automorphism, that is, $xx^{\varphi}x^{\varphi^2}\cdots x^{\varphi^{p-1}} = 1$ for all $x \in G$. If $C_G(A)$ is soluble of derived length $d$, then $G$ is nilpotent of $(p, d, |A|)$-bounded class.

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**Lemma**

The subgroups $C$ and $S$ are invariant under any $FA$-endomorphism $\vartheta$ of $X$.

# Trivialization of $F$

Since there is an $FA$-homomorphism $\xi : X \to G$ with $C, S \leqslant Ker\,\xi$, it is sufficient (and even necessary) to prove that

$[x_1, \ldots, x_{c+1}] \in CS$, where $c$ is the nilpotency class given by EKh-Shumyatsky theorem when $\varphi = 1$.

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By EKh-Shumyatsky theorem, $[x_1, \dots, x_{c+1}] \in CST$,

that is, we need to eliminate $T$.

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An analogue of Higman's lemma gives that we can assume that each $c_i$ depends on all $x_1, \ldots, x_{c+1}$, and on $\varphi$.

One can show that we can furthermore assume that each $c_i$ has the form

$$[[x_{i_1}^{a_*}, \ldots], [x_{i_2}^{a_*}, \ldots], \ldots, [x_{i_{c+1}}^{a_*}, \ldots]] \qquad (a_* \in A),$$

where $\{i_1, i_2, \ldots, i_{c+1}\} = \{1, 2, \ldots, c+1\}$ and there is at least one $\varphi$ among "dots" in at least one of the subcommutators $[x_{i_k}^{a_*}, \ldots]$ .

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And so on, at each step we double the number of occurrences of $\varphi$ in the new $c_i$.

Since $X\langle\varphi\rangle$ is nilpotent (being a finite $p$-group!), in the end we get $\equiv 1$, as required.

# Proof of exponent theorem.

By known results, proof of Theorem 2 reduces to the following result.

**Theorem 2′**

If a finite $p$-group $G$ admits a Frobenius group of automorphisms $FA$ with kernel $F = \langle \varphi \rangle$ of order $p$ and complement $A$ such that $\varphi$ is a splitting automorphism, that is, $xx^{\varphi}x^{\varphi^2}\cdots x^{\varphi^{p-1}} = 1$ for all $x \in G$, then the exponent of $P$ is bounded in terms of the exponent of $C_P(A)$.

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It remains to obtain a bound for the exponent of $V = G/[G, G]$.

# Abelian case: eigenspaces.

Consider $V = G/[G,G]$ as a $\mathbb{Z}FA$-module, with additive notation. In particular, $v + v\varphi + v\varphi^2 + \cdots + v\varphi^{p-1} = 0$ for all $v \in V$ by hypothesis.

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Extend the ground ring by a primitive $p$th root of unity $\omega$, forming $W = V \otimes_{\mathbb{Z}} \mathbb{Z}[\omega]$. Still have $w + w\varphi + w\varphi^2 + \cdots + w\varphi^{p-1} = 0$ for all $w \in W$.

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Define analogues of eigenspaces for the "linear transformation" $\varphi$:

$$W_i = \{w \in W \mid w\varphi = \omega^i w\}.$$

Then $W$ is an "almost direct sum" of the $W_i$:

$$pW \subseteq W_0 + W_1 + \cdots + W_{p-1}$$

and

if $w_0 + w_1 + \cdots + w_{p-1} = 0$ for $w_i \in W_i$, then $pw_i = 0$ for all $i$.

# $A$-orbits.

First: since $\varphi = 1$ on $W_0$, for $x \in W_0$ we have $px = x + x\varphi + \cdots + x\varphi^{p-1} = 0$, so that $pW_0 = 0$.

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$W_i\alpha \subseteq W_{ri}$ for all $i \in \mathbb{Z}/p\mathbb{Z}$. Indeed, if $x_i \in W_i$, then
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Given $u_k \in W_k$ for $k \neq 0$, to lighten the notation we denote $u_k\alpha^i$ by
$u_{r^ik}$; note that $u_{r^ik} \in W_{r^ik}$.

Let $p^e$ be the exponent of $C_G(A)$. Claim: $W_i$ are annihilated by $p^{1+e}$.

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In multiplicative notation, the exponent of $G/[G, G]$ divides $p^{2+e}$, so the exponent of $G$ divides $p^{c(2+e)}$, where $c$ is the nilpotency class of $G$, which is bounded in terms of $p$.

# Remark

If, for some reason, it is known that the derived length $s$ of the group $G$ in Theorems 1 or 2, or in the Corollary, is relatively small, then EKh-81 can be used instead to give a possibly better estimate

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A smaller bound for the nilpotency class would also imply a smaller bound for the exponent.

# Generalizations

In EKh-91 general nilpotency theorem was proved: if a group $G$ admits a group of operators $\Omega$ such that $G\Omega$ is nilpotent, $G$ satisfies $\Omega$-laws which after $\Omega \to 1$ imply nilpotency of class $c$,

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Similarly, the same arguments as above prove

**Theorem 1″**

Suppose that a soluble group $FA$ with normal Sylow $p$-subgroup $F$ and Hall $p'$-subgroup $A$ acts by automorphisms on a finite $p$-group $G$ in such a manner that for some fixed $\varphi_1, \dots, \varphi_p \in F$ we have $x^{\varphi_1} x^{\varphi_2} \cdots x^{\varphi_p} = 1$ for all $x \in G$. If $C_G(A)$ is soluble of derived length $d$, then $G$ is nilpotent of $(p, d, |A|)$-bounded class.

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# Generalizations

There is also local nilpotency theorem in EKh-93, which may also have generalizations in the context of additional group of automorphisms...