Centralizers in Simple Locally Finite Groups

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Locally Finite Groups

A group is called a **locally finite group** if every finitely generated subgroup is a finite group.

Example 1 *Clearly every finite group is a locally finite group.*

In this talk we are mainly interested in infinite locally finite groups.

Locally Finite Groups

Example 2 *Let* $G_1 \leq G_2 \leq G_3 \leq \ldots$ *be an increasing chain of finite groups.*

Then

$$G = \bigcup_{i=1}^{\infty} G_i$$

The group G *is a locally finite group. If we choose*

$G_n = Sym(n)$, *then we obtain the finitary symmetric group*

$$FSym(\mathbf{N}) = \bigcup_{n=1}^{\infty} Sym(n)$$

Example 3 *Let Ω be a set of infinite cardinality κ. Let $Sym(\Omega)$ be the symmetric group on the set Ω. For each $g \in Sym(\Omega)$ define*

$$supp\ (g) = \{\alpha \in \Omega \ : \ \alpha.g \neq \alpha \quad \}$$

Then

$$FSym(\Omega) = \{g \in Sym(\Omega) \ : \quad |supp(g)| < \infty\}$$

is a locally finite group of cardinality κ. Moreover $Alt(\Omega)$ is an infinite simple locally finite group of cardinality κ.

Locally Finite Groups

A field $\mathbf{F}$ is called a **locally finite field** if every finitely generated subfield is a finite field.

Example 4 *Consider the finite field* $\mathbf{Z}_p$. *Let* $\overline{\mathbf{Z}_p}$ *denote the algebraic closure of* $\mathbf{Z}_p$.
The subfields of $\overline{\mathbf{Z}_p}$ *are locally finite fields of characteristic* p.

Locally Finite Groups

Let $\mathbf{F}$ be a locally finite field.

The group of all invertible $n \times n$ matrices over $\mathbf{F}$ is the **General Linear Group** denoted by $GL(n, \mathbf{F})$ and it is a locally finite group. Since homomorphic images and subgroups of locally finite groups are locally finite the projective special linear group $PSL(n, \mathbf{F})$ are also locally finite.

Projective special linear groups over locally finite fields of order greater than or equal to 60 are all simple.

Locally Finite Simple Groups

Recall that a group G is called a **linear group** if there exists a finite dimensional vector space V and a monomorphism

$$\phi : G \to GL(V)$$

The groups $PSL(n, \mathbf{F}), PSU(n, \mathbf{F})$ are examples of linear groups.

A group G is called a **non-linear group** if G does not have a faithful linear representation on a finite dimensional vector space V.
Example of a non-linear group is $Alt(\Omega)$ for an infinite set Ω.

Locally Finite Simple Groups

The simple locally finite groups can be studied in two classes:

(i) The linear simple locally finite groups

(ii) The non-linear simple locally finite groups.

Locally Finite Simple Groups

It is possible by using embedding of finite alternating groups into larger alternating groups to construct uncountably many non-isomorphic, countably infinite, non-linear simple locally finite groups.

Locally Finite Simple Groups of Lie Type

The simple periodic linear groups are characterized:

Theorem 1 *(Belyaev, Borovik, Hartley-Shute and Thomas.)*
If G is an infinite locally finite simple linear group, then there exists a locally finite field F and G is isomorphic to the simple locally finite group of Lie type.
They are the simple Chevalley groups and their twisted versions over locally finite fields.
(See [1], [3], [6], [12].)

Centralizers

In 1954 R. Brauer indicated the importance of the centralizers of involutions in the classification of the finite simple groups. He asked whether it is possible to detect a finite simple group from the structure of the centralizer of its involution. Then it became a program in the classification of finite simple groups.

Centralizers

There were two types of questions:

1) Given the finite simple group G. Find the structure of $C_G(i)$ for all involutions $i \in G$.

2) Find the structure of the simple group G when the group $H = C_G(i)$ is known for an involution $i \in G$.

Centralizers

One may ask similar questions for the locally finite, simple groups or LFS-groups as we will call them.

One may generalize these questions from the centralizers of involutions to the centralizers of arbitrary elements or subgroups.

Centralizers

$1'$) Given an infinite LFS-group, find the structure of the centralizers of elements (or subgroups) in particular centralizers of involutions.

$2'$) Given the structure of the centralizer of an element in a LFS-group G, find the structure of the LFS-group G.

Kegel's question

In this respect the following question is asked by O. H. Kegel.

Question. (Otto H. Kegel; Unsolved problems in Group Theory Kourovka Notebook 5th issue question 5.18 (1976)).

Let G be an infinite locally finite simple group. Is $|C_G(g)|$ infinite for every element $g \in G$.

An affirmative answer is given to this question.

Centralizers

Theorem 2 *(Hartley -Kuzucuoğlu Proc. London Math. Soc. 1991)*

In an infinite locally finite simple group, the centralizer of every element is infinite.

Simple Locally Finite Groups of Lie Type

The connection between simple locally finite groups and simple finite groups are formed by the following notion.

A *Kegel cover* $\mathcal{K}$ of a locally finite group G is a set $\mathcal{K} = \{(H_i, M_i) \mid i \in I\}$ such that for all $i \in I$ the group H_i is a finite subgroup of G, M_i is a maximal normal subgroup of H_i and for each finite subgroup K of G, there exists $i \in I$ such that $K \leq H_i$ and $K \cap M_i = 1$.
The simple groups H_i/M_i are called **Kegel factors** of $\mathcal{K}$.
Kegel proved that:

Simple Locally Finite Groups of Lie Type

Theorem 3 *(O. H. Kegel [7],[8, Lemma 4.5].) Every infinite simple locally finite group has such a cover.*

If order of G is countably infinite, then we have an ascending sequence of finite subgroups $G_1 \leq G_2 \ldots G_i \leq G_{i+1} \ldots$ where $G = \bigcup G_i, \quad N_i$ maximal normal in G_i and $G_i \cap N_{i+1} = 1$.

The sequence $\{(G_i, N_i) \ : \ i \in \mathbf{N}\}$ is called a **Kegel sequence**.

Kegel Sequence

The following two articles give examples of simple locally finite groups such that for every Kegel sequence we can <u>NOT</u> take $N_i = 1$.

A.E. Zalesskii, V.N. Serezhkin Finite Linear Groups Generated by Reflections. Math. USSR: IZV. 17 (1981), 477-503.

K. Hickin Universal locally finite central extensions of groups. Proc. London Math. Soc. (1986), 53-72.

Simple Locally Finite Groups of Lie Type

For the study of centralizers of elements in infinite LFS-groups one of the obstacle is the following. We know the structure of the centralizer of subgroups or elements X in the finite simple section H_i/M_i and in general $C_{H_i/M_i}(X)$ is not isomorphic or equal to $C_{H_i}(X)M_i/M_i$, for this reason the information about the centralizers in the finite simple group, in general does not transform directly to the information on $C_G(X) = \bigcup_{i \in I} C_{H_i}(X)$.

Generalization of Kegel's question

Then the natural question:

Question. Is the centralizer of every finite subgroup in a countably infinite simple locally finite group infinite?

The answer to this question is negative as the following easy example shows.

Example 5 *One can see easily that in $PSL(2, \mathbf{F})$ where the characteristic of the field $Char(\mathbf{F})$ is odd and $\mathbf{F}$ is algebraically closed.*
Let

$$A = \langle \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} Z, \ \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} Z \mid \lambda^2 = -1 \rangle$$

A is an abelian subgroup of order 4 and $C_{PSL(2,\mathbf{F})}(A) = A$. Hence the answer to the question is ***not*** *affirmative in general.*

Example 6 *Let $G = PSL(3, F)$ where F is an algebraically closed field of characteristic different from 3.*

Let

$$x = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} Z, \quad y = \begin{pmatrix} 0 & 0 & 1 \\ \alpha^2 & 0 & 0 \\ 0 & \alpha & 0 \end{pmatrix} Z \text{ where } \alpha^3 = 1.$$

$\langle x, y \rangle \cong \mathbf{Z}_3 \times \mathbf{Z}_3$. Then $C_G(\langle x, y \rangle) = \langle x, y \rangle$

Locally Finite Simple Groups

In fact for non-abelian finite subgroups, one can give more examples. Moreover for any simple linear group one can find a finite subgroup with trivial centralizer.

Locally Finite Groups

Question (Kleidman-Wilson) [9], Archiv Math. (1987):
”Classify all simple locally finite groups in which every proper subgroup is almost locally soluble?”

Locally Finite Groups

They proved:

Theorem 4 *Every infinite simple locally finite minimal non-almost locally soluble group is isomorphic to* $PSL(2, \mathbf{F})$ *or* $Sz(\mathbf{F})$ *for some locally finite field* $\mathbf{F}$ *which has no infinite proper subfields.*

In particular such groups are linear.

In this spirit we will ask the following:

Question. Classify all infinite simple locally finite groups in which centralizer of an involution is locally soluble by finite.

Centralizers of involutions

Question. Does there exist a simple locally finite group in which centralizer of every involution is soluble by finite?

Locally Finite Groups

The examples of Meierfrankenfeld shows that, in these groups centralizer of every involution is almost locally soluble. So there are non-linear LFS-groups in which centralizer of every involution (element) is almost locally soluble. In order to get rid of non-linear LFS-groups for this question we restrict the question for those groups G which has a Kegel sequence $\{(G_i, 1) \mid i \in \mathbf{N}\}$.
Then one can extract the following theorem from[2, Theorem 4].

Theorem 5 *(Berkman-Kuzucuoğlu-Özyurt) Let* G *be an infinite simple locally finite group centralizer of an involution has a locally soluble subgroup of finite index. If* G *has a Kegel cover* $\mathcal{K}$ *such that* $(H_i, M_i) \in \mathcal{K}, \quad M_i/O_{2'}(M_i)$ *is hypercentral in* $H_i/O_{2'}(M_i)$*. Then* G *is isomorphic to one of the following:*
(i) $PSL(2, \mathbf{K})$ *where* $\mathbf{K}$ *is an infinite locally finite field of arbitrary characteristic,* $PSL(3, \mathbf{F})$*,* $PSU(3, \mathbf{F})$ *and* $Sz(\mathbf{F})$ *where* $\mathbf{F}$ *is an infinite locally finite field of characteristic* 2*. In this case, centralizers of involutions are soluble.*
(ii) *There exist involutions* $i, j \in G$ *such that* $C_G(i)$ *is locally soluble by finite and* $C_G(j)$ *involves an infinite simple group if and only if* $G \cong PSp(4, \mathbf{F})$ *and the characteristic of* $\mathbf{F}$ *is* 2*.*

Corollary 6 *If G is as in the above Theorem and if we assume that centralizer of every involution has a locally soluble subgroup of finite index, then G is as in* **(i)** *in the above Theorem.*

Locally Finite Simple Groups of Lie Type

Then the following question remains open for non-linear groups:

Question (Hartley) Is the centralizer of every finite subgroup in a non-linear countably infinite simple locally finite group infinite?

Locally Finite Simple Groups

Stronger question. Does the centralizers of elements in non-linear locally finite simple groups contains infinite abelian subgroups which has elements of order p_i for infinitely many distinct prime p_i.

The answer to this question in this generality is negative by the construction of Meierfrankenfeld; Locally Finite Simple Group with a p-group as centralizer, Turkish J. Math. 31 Supplementary Edition (2007).

Locally Finite Simple Groups

Theorem 7 *(Meierfrankenfeld) Let* π *be a non-empty set of primes. Then there exists a non-linear simple locally finite group* G *such that centralizer of every* π*- element in* G *is almost locally solvable* π*-group. There exists an element whose centralizer is locally soluble* π*-group.*
In particular if $\pi = \{p\}$ *centralizer of an element is a* p*-group.*

Semisimple Subgroups

Definition 1 . *Let G be a simple group of Lie type. A finite subgroup A of G is called a* **semisimple subgroup** *if every element of A is a semisimple element in G.*

Recall that an element in a simple group of Lie type is **semisimple** if the order of the element and the characteristic of the field is relatively prime. In the alternating groups all elements are semisimple.

Locally Finite Simple Groups

Definition. Let G be a countably infinite simple locally finite group and F be a finite subgroup of G. The group F is called $\mathcal{K}$-**semisimple** subgroup of G, if G has a Kegel sequence $\mathcal{K}= \{(G_i, M_i) : \quad i \in \mathbf{N} \}$ such that $(|M_i|, |F|) = 1$, M_i are soluble for all i and if G_i/M_i is a linear group over a field of characteristic p_i, then $(p_i, |F|) = 1$.

Centralizers in Locally Finite Simple Groups

Theorem 8 *(K. Ersoy, M. Kuzucuoğlu, 2012) Let* G *be a non-linear simple locally finite group which has a Kegel sequence* $\mathcal{K}= \{(G_i, 1) : \ \ i \in \mathbf{N} \ \}$ *consisting of finite simple subgroups. Then for any finite* $\mathcal{K}$*-semisimple subgroup* F*, the centralizer* $C_G(F)$ *is an infinite group.*
Moreover $C_G(F)$ *has an infinite abelian subgroup* A *isomorphic to the restricted direct product of* $\mathbf{Z}_{p_i}$ *for infinitely many distinct prime* p_i*.*

Locally Finite Simple Groups

Theorem 9 *(K. Ersoy-M. Kuzucuoğlu, 2012) Let* G *be a simple locally finite group which is a direct limit of finite alternating groups, and* F *be a finite subgroup of* G. *Then* $C_G(F)$ *contains an abelian subgroup* A *isomorphic to* $Dr_{p_i}\,\mathbf{Z}_{p_i}$ *for infinitely many distinct primes* p_i.

Semisimple Subgroup

We call a subgroup F in a LFS-group G with a Kegel sequence $\mathcal{K}$ is semisimple if FN_i/N_i is a semisimple subgroup in the finite simple group G_i/N_i for all $i \in \mathbf{N}$.

Bound on the index of Centralizers

Lemma 10 *Let G be a finite group and $N \trianglelefteq G$. Let F be a subgroup generated k elements. Then*

$$|C_{G/N}(FN/N) : C_G(F)N/N| \leq |C_N(F)||N|^{k-1}$$

Centralizers of Semisimple Subgroups

Proposition 11 *Let G be a countably infinite non-linear LFS-group with a Kegel sequence $\mathcal{K} = \{(G_i, N_i) \mid i \in \mathbf{N}\}$. If there exists an upper bound for $\{|N_i| \mid i \in \mathbf{N}\}$, then for any finite semisimple subgroup F in G the subgroup $C_G(F)$ has elements of order p_i for infinitely many distinct prime p_i. In particular $C_G(F)$ is an infinite group.*

THANK YOU

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