SOME QUESTIONS ARISING FROM THE STUDY OF THE GENERATING GRAPH

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The graph $\Gamma(G)$ has “many” edges when $G$ is a finite non abelian simple group: a finite non abelian simple group $G$ can be generated by 2 elements and the probability that a random pair of vertices of $\Gamma(G)$ is connected by an edge tends to 1 as $|G|$ tends to infinity.
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Many deep results about finite simple groups $G$ can equivalently be stated as theorems about $\Gamma(G)$.

- (Guralnick and Kantor, 2000) There is no isolated vertex in $\Gamma(G)$.
- (Breuer, Guralnick and Kantor, 2008) The diameter of $\Gamma(G)$ is at most 2.
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**Theorem (Breuer, Guralnick, AL, Maróti, Nagy (2010))**

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**Theorem (Breuer, Guralnick, Al, Maróti, Nagy (2010))**

For every sufficiently large finite simple group $G$, the graph $\Gamma(G)$ contains a Hamiltonian cycle.

**Conjecture**

Let $G$ be a finite group with at least 4 elements. Then $\Gamma(G)$ contains a Hamiltonian cycle if and only if $G/N$ is cyclic for all non-trivial normal subgroups $N$ of $G$. 
The conjecture is true for finite soluble groups.

For every sufficiently large symmetric group $\text{Sym}(n)$, the graph $\Gamma(\text{Sym}(n))$ contains a Hamiltonian cycle.

For every sufficiently large non-abelian finite simple group $S$, the graph $\Gamma(S \wr C_m)$ contains a Hamiltonian cycle, where $m$ denotes a prime power.
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For every sufficiently large non-abelian finite simple group $S$, the graph $\Gamma(S \wr C_m)$ contains a Hamiltonian cycle, where $m$ denotes a prime power.

Moreover computer calculations show that the following groups contain Hamiltonian cycles.

- Non-abelian simple groups of orders at most $10^7$,
- groups $G$ containing a unique minimal normal subgroup $N$ such that $N$ has order at most $10^6$, $N$ is nonsolvable, and $G/N$ is cyclic,
- alternating and symmetric groups on $n$ points, with $5 \leq n \leq 13$,
- sporadic simple groups and automorphism groups of sporadic simple groups.
The proofs rely on classical results in graph theory that ensure that a graph $\Gamma$ contains a Hamiltonian cycle provided that “many vertices of $\Gamma$ have large degree”.
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**Pósa’s criterion**

A graph with $m$ vertices and list of vertex degrees $d_1 \leq \ldots \leq d_m$ contains a Hamiltonian cycle if $d_k \geq k + 1$ for all $k < m/2$. 
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The \( n \)-closure of a graph \( \Gamma \) is the graph obtained from \( \Gamma \) by recursively joining pairs of nonadjacent vertices whose degree sum is at least \( n \) until no such pair remains.

**Bondy, Chvátal**

A graph with \( m \) vertices is Hamiltonian if and only if its \( m \)-closure is Hamiltonian.
Let $m \in \mathbb{N}$ and let $S$ be a nonabelian simple group and consider the wreath product $G = S \wr C_m$. Is the generating graph of $G$ Hamiltonian?
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- Let $\pi : G = S \wr C_m \to C_m$ the projection to the top group $C_m$.
- If $\pi(g_1)$ and $\pi(g_2)$ generate $C_m$, then $g_1$ and $g_2$ are connected in $\Gamma(G)$ with high probability.
- $\deg(g, \Gamma(G)) \sim \deg(\pi(g), \Gamma(C_m))|S|^m$ if $S$ is large.
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The conditions in Posa’s criterion and in the Bondy-Chvátal Theorem are satisfied by $\Gamma(G)$ provided that the graph $\Gamma(C_m)$ satisfies similar conditions.
**Definition**

\[ \Lambda_m := \text{the } (m + 1)-\text{closure of the generating graph } \Gamma(C_m). \]

We say that \( m \) is **Hamiltonian** if every \( 1 \neq x \in C_m \) generating a subgroup of \( C_m \) of odd index is connected in \( \Lambda_m \) to any other vertex.
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**Theorem (E. Crestani and AL (2011))**

Assume that \( m \) is Hamiltonian. There exists a positive integer \( \tau \) such that if \( S \) is a nonabelian simple group with \( |S| \geq \tau \), then the graph \( \Gamma(S \wr C_m) \) contains a Hamiltonian cycle.
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**Conjecture**

Every positive integer is Hamiltonian.
Corollary

Let $m = \prod_{i=1}^{s} p_i^{\alpha_i}$, where $p_s < p_{s-1} < \cdots < p_1$ are distinct primes and $\alpha_i > 0$ for every $1 \leq i \leq s$. Assume that one of the following holds:

1. $s \leq 2$;
2. $\varphi(m)/m \geq (p_s - 1)/(2p_s - 1)$;
3. $m$ is odd and $p_s \geq s + 1$;
4. $m$ is even and $p_{s-1} \geq 2s - 1$.

There exists $\tau$ such that if $|S| \geq \tau$, then the graph $\Gamma(S \wr C_m)$ contains a Hamiltonian cycle.
S a finite non abelian simple group

\( \delta(S) := \) number of \((\text{Aut } S)\)-orbits on pairs of generators for \( S \)

\( S^n \) is still 2-generated if and only if \( n \leq \delta(S) \).
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$\delta(S)$ increases with $|S|$ and the probability that a random pair of vertices of $\Gamma(S^{\delta(S)})$ is connected by an edge tends to 0 as $|S| \to \infty$.

Nevertheless, the abundance of edges in the graph $\Gamma(S)$ reflects on $\Gamma(S^{\delta(S)})$. 
Let $n \leq \delta(S) = \delta$. If $n$ is large, it is no more true that $\Gamma(S^n)$ has no isolated vertices. We will concentrate our attention on the subgraph $\Gamma_n(S)$ obtained from $\Gamma(S^n)$ by removing the isolated vertices.
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- $\{(a_1, b_1), \ldots, (a_\delta, b_\delta)\}$ a set of representatives for the $\text{Aut}(S)$-orbits on the ordered pairs of generators for $S$
- $C_1, \ldots, C_u$ the $\text{Aut}(S)$-orbits on $S \setminus \{1\}$
- $\delta_r = |\{i \mid a_i \in C_r\}|$
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\[
S^n = \langle (x_1, \ldots, x_n), (y_1, \ldots, y_n) \rangle \quad \iff \\
(x_1, y_1), \ldots, (x_n, y_n) \text{ are not } \text{Aut}(S)\text{-conjugated generating pairs of } S
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$x = (x_1, \ldots, x_n)$ is a nonisolated vertex of $\Gamma(S^n)$

$$\iff |\{i \mid x_i \in C_r\}| \leq \delta_r \text{ for } 1 \leq r \leq u.$$
The generating graph
Hamiltonian cycles
Direct power of simple groups
Soluble groups

$S = \text{Alt}(5), \quad \delta(S) = 19, \quad u = 3$

$(1, 2, 3, 4, 5) \in C_1 \Rightarrow \delta_1 = 10$
$(1, 2, 3) \in C_2 \Rightarrow \delta_2 = 6$
$(1, 2)(3, 4) \in C_3 \Rightarrow \delta_3 = 3$

$x = (x_1, \ldots, x_n)$
is a non-isolated vertex in $\Gamma(S^n)$
if and only if $x$ has
at most 10 entries of order 5,
at most 6 entries of order 3,
at most 3 entries of order 2.
\begin{itemize}
  \item \( \text{Aut}(S^\delta) \cong \text{Aut}(S) \wr \text{Sym}(\delta) \leq \text{Aut}(\Gamma_{\delta}(S)). \)
  \item \( S^\delta = \langle x, y \rangle = \langle \bar{x}, \bar{y} \rangle \Rightarrow (\bar{x}, \bar{y}) = (x^\alpha, y^\alpha) \ \exists \alpha \in \text{Aut}(S^\delta).\)
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  \item $\Gamma_{\delta}(S)$ is vertex-transitive and edge-transitive.
  \item $\Gamma_{\delta}(S)$ has $\frac{|\text{Aut}(S)|^\delta \delta!}{2}$ edges and $\frac{|\text{Aut}(S)|^\delta \delta!}{\prod_{1 \leq i \leq u} \gamma_i^{\delta_i} \delta_i!}$ vertices, with $\gamma_i = |C_{\text{Aut}(S)}(x_i)|$ for $x_i \in C_i$.
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For example, \Gamma_{19}(Alt(5)) \text{ is a graph with } 2^{45} \cdot 3^{14} \cdot 5^9 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \text{ vertices, whose degree is equal to } 2^{28} \cdot 3^{13} \cdot 5^{13} \cdot 7.
Theorem (E. Crestani and AL (2011))

If $S$ is a non abelian simple group, then the graph $\Gamma_n(S)$ is connected for $n \leq \delta(S)$. 
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Let $S$ be a finite non abelian simple group and let $\delta = \delta(S)$.

1. If $n < \delta$, then the diameter of $\Gamma_n(S)$ is at most $2 + 4(n - 1)$.
2. The diameter of $\Gamma_\delta(S)$ is at most $4\delta + c$ for an absolute constant $c$.
3. The diameter of $\Gamma_\delta(S)$ is at most $4\delta$ if $|S|$ is large enough.
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**Theorem (E. Crestani and AL (2011))**

Let $S = SL(2, 2^p)$ with $p$ an odd prime. Then $\text{diam}(\Gamma_\delta(S)) \geq 2^{p-2} - 1$ if $p$ is large enough.
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If a graph has “many” edges, then by Turán’s Theorem, it should contain a “large” complete subgraph. Applying this result to $\Gamma(S)$ when $S$ is a nonabelian simple group with large order, it follows:

**Theorem (Liebeck and Shalev (1995))**

There exists a positive constant $c_1$ such that $c_1 \cdot m(S) \leq \omega(\Gamma(S))$ for any finite simple group $S$ where $m(S)$ denotes the minimal index of a proper subgroup in $S$. 
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The value of $\omega(\Gamma(S))$ is in general much larger than $m(S)$. For example $\omega(\Gamma(\text{Alt}(n))) = 2^{n-2}$ if $n$ is large and not divisible by 4.
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**Theorem (A. Maroti and AL (2009))**

If \( \delta = \delta(S) \), then \( \omega(\Gamma(S^\delta)) = \omega(\Gamma_\delta(S)) \) is at most \((1 + o(1))m(S)\), so \( \omega(\Gamma_i(S)) \) decreases drastically with \( i \).
No good lower bound for $\omega(\Gamma_\delta(S))$ is known.
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**Question**

Do there exist simple groups $S$ for which $\omega(\Gamma_\delta(S))$ is arbitrarily large?
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- $G = \langle x, y \rangle \Rightarrow \{x, y, xy\}$ is a complete subgraph of $\Gamma(G)$.
- If $G$ is 2-generated, then $\omega(\Gamma(G)) \geq 3$. 
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Is there a non abelian simple group $S$ for which $\omega(\Gamma_\delta(S)) > 3$?
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**Question**

Is there a non abelian simple group $S$ for which $\omega(\Gamma_\delta(S)) > 3$?

**Theorem**

$\omega(\Gamma_{19}(\text{Alt}(5))) = 4$. 

Andrea Lucchini

Some questions arising from the study of the generating graph
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A set that generates $G$ pairwise cannot contain two elements of any proper subgroup, hence $\omega(\Gamma(G)) \leq \sigma(G)$. 
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In general $\omega(\Gamma(G)) \neq \sigma(G)$. $\omega(\Gamma(\text{Alt}(5))) = 8$ and $\sigma(\text{Alt}(5)) = 10$.

However no example is known of a finite 2-generated soluble non cyclic group $G$ with $\omega(\Gamma(G)) \neq \sigma(G)$. 
If $G$ is a finite, 2-generated, non cyclic, soluble group and $A$ is the set of the chief factors $G$ having more than one complement, then

$$
\min_{A \in \mathcal{A}} (1 + |\text{End}_G(A)|) \leq \omega(\Gamma(G)) \leq \sigma(G) = \min_{A \in \mathcal{A}} (1 + |A|).
$$
**Theorem (E. Crestani and AL (2012))**

*If* $G$ *is a finite, 2-generated, non cyclic, soluble group and* $A$ *is the set of the chief factors* $G$ *having more than one complement, then*

$$\min_{A \in A} (1 + |\text{End}_G(A)|) \leq \omega(\Gamma(G)) \leq \sigma(G) = \min_{A \in A} (1 + |A|).$$

**Corollary (A. Maroti and AL (2009))**

*Let* $G$ *be a finite soluble group with Fitting height at most 2. Then* $\omega(\Gamma(G)) = \sigma(G)$. 
**Definition**

\[ \mu_d(G) := \text{the largest } m \text{ for which there exists an } m\text{-tuple of elements of } G \text{ such that any of its } d \text{ entries generate } G \quad (\mu_2(G) = \omega(\Gamma(G))). \]

**Theorem**

Let \( G \) be a \( d \)-generated finite soluble group with \( d \geq 2 \) and let \( A \) be the set of the chief factors \( G \) having more than one complement. Assume that a positive integer \( t \) satisfies the following property:

1. \( t \leq |A| \) for each \( A \in A \) with \( C_G(A) = G \).
2. \( \left( \frac{t-1}{d-1} \right) \leq |\text{End}_G(A)| \) for each \( A \in A \) with \( C_G(A) \neq G \).

Then \( \mu_G(d) \geq t \).

**Corollary**

Let \( G \) be a \( d \)-generated finite group, with \( d \geq 2 \), and let \( p \) be the smallest prime divisor of the order of \( G \). Then \( \left( \frac{\mu_d(G)}{d-1} \right) > p \).
A question in linear algebra plays a crucial role in the study of the value of $\mu_d(G)$ when $G$ is soluble.

$M_{r \times s}(F)$ the ring of the $r \times s$ matrices with coefficients over $F$.

Let $t \geq d$. Assume that $A_1, \ldots, A_t \in M_{n \times n}(F)$ have the property that

$\text{rank} \left( A_{i_1} \cdots A_{i_d} \right) = n$ whenever $1 \leq i_1 < i_2 < \cdots < i_d \leq t$.

Can we find $B_1, \ldots, B_t \in M_{n(d-1) \times n}(F)$ with the property that

$\det \begin{pmatrix} A_{i_1} & \cdots & A_{i_d} \\ B_{i_1} & \cdots & B_{i_d} \end{pmatrix} \neq 0$ whenever $1 \leq i_1 < i_2 < \cdots < i_d \leq t$?
PROOF.

\[
\begin{align*}
    x &= (x_1, \ldots, x_\delta) \quad \bar{x} = (\bar{x}_1, \ldots, \bar{x}_\delta) \\
    y &= (y_1, \ldots, y_\delta) \quad \bar{y} = (\bar{y}_1, \ldots, \bar{y}_\delta) \\
    \downarrow \\
    (\bar{x}_{i\pi}, \bar{y}_{i\pi}) &= (x_i)_{a_i}^{\pi} \quad \exists \pi \in \text{Sym}(\delta), (a_1, \ldots, a_\delta) \in \text{Aut}(S)^\delta \\
    \downarrow \\
    (\bar{x}, \bar{y}) &= (x^\alpha, y^\alpha) \quad \text{for } \alpha = (a_1, \ldots, a_\delta)\pi \in \text{Aut}(S^\delta).
\end{align*}
\]
**Turán’s Theorem**

Let $\Gamma$ be a graph with $n$ vertices and $e$ edges. If $\omega(\Gamma) \leq r$ then

$$e \leq \left(1 - \frac{1}{r}\right) \frac{n^2}{2}.$$
Assume that \( \{g_1, \ldots, g_r\} \) is a complete subgraph of \( \Gamma_\delta(S) \).

\( S^\delta = \langle g_1, g_2 \rangle \) and for each \( i \in \{3, \ldots, r\} \) there exists a word \( w_i(x_1, x_2) \) such that \( g_i = w_i(g_1, g_2) \).

\[
S = \langle s_1, s_2 \rangle \\
\downarrow \\
\{s_1, s_2, w_3(s_1, s_2), \ldots, w_r(s_1, s_2)\}
\]

is a complete subgraph of \( \Gamma(S) \).